# Math 2184 §10 Fall 2020
# Linear Algebra I Mastery Quiz 13
# Due Midnight on Thursday, December 10

This week's mastery quiz has twelve topics. **Do not answer all twelve questions.** You may answer up to *three* topics in total. You should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

21. Orthogonal Decomposition
20. Inner Products
19. Dot Product and Projection
17. Similarity and Trace
16. Change of Basis
15. Complex and Generalized Eigenvectors
14. Characteristic Polynomials and Finding Eigensystems
13. Eigenvectors and Determinants
12. Matrices of Linear Transformations
11. Bases and Coordinates
10. Vector Space Linear Transformations
9. Vector Spaces and Subspaces

21. **Orthogonal Decomposition** Let $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}$.

(a) Find an orthogonal basis for $U$.

(b) Find a basis for $U^{\perp}$.

(c) Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -3 \end{bmatrix}$. Find the orthogonal decomposition of $\mathbf{y}$ with respect to $U$.

20. **Inner Products**

Let $V = M_2$ be the space of $2 \times 2$ matrices and define $\langle A, B \rangle = \mathrm{Tr}(A^T B)$ where Tr is the trace.

(a) Prove that this is an inner product.

(b) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$. Compute $\|A\|$ and $\|B\|$.

(c) Compute $\mathrm{Proj}_A\, B$ and the projection of $B$ orthogonal to $A$.

19. **Dot Product and Projection** Let $\mathbf{u} = (3, 0, -4)$ and $\mathbf{v} = (-2, 2, 1)$.

(a) Compute $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

(b) Compute $\text{Proj}_{\mathbf{v}}\,\mathbf{u}$ and the projection of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

(c) What is the angle between $\mathbf{u}$ and $\mathbf{v}$?

(d) Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

17. **Similarity and Trace**

(a) Let $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. Show that $A$ is similar to $B$.

(b) Let $C = \begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Show that neither matrix is similar to $A$.

16. **Change of Basis**

Let

$$E = \left\{ \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} \right\}, \qquad F = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Find the transition matrix from $E$ to $F$.

15. **Complex and Generalized Eigenvectors**

(a) Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Find all (complex!) eigenvalues of $A$ and find a (possibly complex) eigenvector for each eigenvalue.

(b) Let $B = \begin{bmatrix} 1 & 11 & -7 \\ 5 & -1 & 7 \\ 2 & -6 & 7 \end{bmatrix}$. Find an eigenvector of eigenvalue 0, and then find a generalized eigenvector of eigenvalue 0.

14. **Characteristic Polynomials and Finding Eigensystems**

Let $A = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. Find the characteristic polynomial, the eigenvalues, and a basis for each associated eigenspace.

13. **Eigenvectors and Determinants**

(a) Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. Show that $(1, -1, 0)$ is an eigenvector of $A$. What is the corresponding eigenvalue?

(b) Compute $\det \begin{bmatrix} 2 & 3 & -2 \\ 4 & -3 & 7 \\ 2 & 1 & 5 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & 3 & 4 & 0 \\ 3 & 2 & 1 & 3 \\ 2 & 1 & 4 & 0 \\ -1 & 3 & -2 & 0 \end{bmatrix}$

12. **Matrices of Linear Transformations**

(a) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(x, y, z) = (x + y, 3x - 2y + z, y - 2z)$. Give a matrix for $L$ with respect to the bases to $E = \{(2, 1, -1), (0, 3, 1), (4, -2, 0)$ and $F = \{(1, 1, 1), (1, 0, 1), (1, 0, 0)\}$.

(b) Let $T : \mathcal{P}_2(x) \to \mathbb{R}^2$ be defined by $T(f) = \begin{bmatrix} f(3) - f(1) \\ f(2) + f(-2) \end{bmatrix}$. Find a matrix for $T$ with respect to the basis $E = \{1, x, x^2\}$ and $F = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

11. **Bases and Coordinates**

(a) Prove that $\{1+x, 3-x+x^2, x^2-x\}$ is a basis for $\mathcal{P}_2(x)$. (Please explicitly use the formal definition of a basis.)

(b) Let $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find $\begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}_B$.

10. **Vector Space Linear Transformations**

(a) Let $L : \mathcal{P}_2(x) \to S$, where $S = \{\dots, x_{-1}, x_0, x_1, \dots\}$ be defined by $L(f) = \{\dots, f(-1), f(0), f(1), \dots\}$. Is $L$ a linear function? Prove your answer.

(b) Let $T : \mathbb{R}^2 \to S$ be defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \{\dots, a+b-1, a+b, a+b+1, \dots\}$$

is $T$ a linear function? Prove your answer.

9. **Vector Spaces and Subspaces**

(a) Let $V = \mathcal{P}(x)$ be the space of polynomials, and $U = \{f : f'(3) = 0$. Is $U$ a subspace of $V$? Prove your answer.

(b) Let $V = \mathcal{P}(x)$ be the space of polynomials, and $U = \{f : f'(0) = 3\}$. Is $U$ a subspace of $V$? Prove your answer.