Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 2
Due Noon on Tuesday, September 15

This week's mastery quiz has four topics. Please answer the problems on the new topics, labeled 4 and 3. You may answer *one* of the other two topics if you did not get a "mastery" grade on it last week.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

4. Linear Transformations
3. Linear Independence
2. Vector Equations and Spanning
1. Systems of Linear Equations

4. **Linear Transformations** Consider the following functions:

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+y \\ y-z \\ x+z \end{bmatrix} \qquad g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y-z \\ z^2 \end{bmatrix}$$

(a) One of these two functions is linear. Which one is linear, and why?

Solution: We can check directly that f is linear, by checking that

$$f\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + f\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right)$$

and

$$cf\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = f\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right).$$

In this case it might be easier to observe that g clearly isn't linear. We can see that $g(0,0,0) = (1,0,0) \neq \mathbf{0}$, or observe that $cg(x,y,z) = (cx+c, cy-cz, cz^2) \neq (cx+1, cy-cz, c^2z^2) = g(cx,cy,cz)$, or see that

$$g\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + g\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+2 \\ y_1+y_2-z_1-z_2 \\ z_1^2+z_2^2 \end{bmatrix} \neq \begin{bmatrix} x_1+x_2+1 \\ y_1+y_2-z_1-z_2 \\ (z_1+z_2)^2 \end{bmatrix} = g\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right).$$

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why? **Solution:** Row reducing this matrix gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This means that the columns are linearly independent, so the function is one-to-one; and there is a solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^3$, and thus the function is onto.

(I would probably accept just observing that it reduces to the identity, and thus is both one-to-one and onto.)

3. **Linear Independence**

(a) *Using the formal definition of linear independence*, show that the set

$$\left\{ \mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$$

is linearly independent.

Solution: Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. We can set this up as

$$a\begin{bmatrix}5\\1\\-1\end{bmatrix} + b\begin{bmatrix}1\\2\\-3\end{bmatrix} + c\begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{aligned}5a + b + 2c &= 0\\ a + 2b + c &= 0\\ -a - 3b + 4c &= 0.\end{aligned}$$

This is a homogeneous system of linear equations, so we can set up a matrix row reduction

$$\begin{bmatrix}5 & 1 & 2\\ 1 & 2 & 1\\ -1 & -3 & 4\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

so we see the only solution is $a = b = c = 0$. Thus by definition, the vectors are linearly independent.

(b) Find a linear dependence relationship among the following vectors. That is, write one as a linear combination of the others.

$$\left\{\mathbf{x} = \begin{bmatrix}2\\4\\1\end{bmatrix}, \mathbf{y} = \begin{bmatrix}1\\-3\\2\end{bmatrix}, \mathbf{z} = \begin{bmatrix}6\\2\\6\end{bmatrix}\right\}$$

Solution: We write down the matrix

$$\begin{bmatrix}2 & 1 & 6\\ 4 & -3 & 2\\ 1 & 2 & 6\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}.$$

This tells us that $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ for $a = b = -2c$. Thus in particular, we have $2\mathbf{x} + 2\mathbf{y} - \mathbf{z} = 0$, so $\mathbf{z} = 2\mathbf{x} + 2\mathbf{y}$.

2. **Vector Equations and Spans**

(a) Is the vector $\mathbf{v} = \begin{bmatrix}3\\0\\3\end{bmatrix}$ in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$?

Solution: Yes. We set up the equation

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\0\\3\end{bmatrix}$$

which gives the matrix

$$\begin{bmatrix}1 & -1 & 3\\ 2 & 1 & 0\\ 3 & 0 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}$$

which is consistent So $\mathbf{v}$ is in the span of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(b) Write the vector $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors

$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$

Solution: $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = -\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$

1. **Systems of Linear Equations**

Use row reduction to find all solutions to each system of equations.

(a)

$$\begin{aligned} 3x + y + 2z &= 1 \\ x + 3y + 2z &= -1 \\ 4x + 4y + 4z &= 0 \end{aligned}$$

Solution: $x = 1/2 - z/2, y = -1/2 - z/2$, or $\{(1/2 - z/2, -1/2 - z/2, z)\}$ or

$$\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

(b)

$$\begin{aligned} 2x + 1y - z &= 2 \\ x + 5y + 2z &= 1 \\ -3x - 2y + z &= 3 \end{aligned}$$

Solution: $x = -20, y = 15, z = -27$