Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 3
Due Midnight on Thursday, September 24

This week's mastery quiz has six topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 6 and 5. You may answer *one* of the other four topics if you did not get a "mastery" grade on it already.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file.* You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

6. Matrix Inverses
5. Matrix Multiplication
4. Linear Transformations
3. Linear Independence
2. Vector Equations and Spanning
1. Systems of Linear Equations

6. **Matrix Inverses**

(a) Find the inverse of $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & -2 \\ 0 & 1 & -4 & 1 & 0 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & -2 & -1 \\ 0 & 1 & 0 & -3 & 4 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

so $A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -3 & 4 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

(b) Which of the following matrices are invertible? Justify your answers.

$$\begin{bmatrix} 3 & 1 & 2 \\ 5 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 4 \\ 2 & 2 & 3 \end{bmatrix}$$

Solution: The first isn't invertible since it isn't square. The second isn't invertible becuase the columns are linearly dependent; the third is invertible because the columns are linearly independent.

The fourth we might actually need to row reduce. We get

$$\begin{bmatrix} 1 & 5 & 3 \\ 2 & 1 & 4 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 3 \\ 0 & -9 & -2 \\ 0 & -8 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 3 \\ 0 & -9 & -2 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -11 \end{bmatrix}$$

thus the nullspace is trivial and the columns span $\mathbb{R}^n$ and so the matrix is invertible. Alternatively, we can clearly row reduce this to get the identity, so the matrix is invertible.

5. **Matrix Multiplication**

Compute, or write that the computation is undefined:

(a) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 1 & 4 & 4 \end{bmatrix} =$

Solution: $\begin{bmatrix} 8 & 10 & 13 \\ 14 & 12 & 18 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -2 & 1 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} =$ **Solution:** Undefined

(c) $\begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 4 & -2 & 1 \end{bmatrix} =$

Solution: $\begin{bmatrix} 4 & 8 & 2 \\ 13 & -9 & 3 \\ 20 & -10 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 3 & 0 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 0 & 5 \end{bmatrix} =$

Solution: $\begin{bmatrix} -7 & 7 \\ 18 & 7 \end{bmatrix}$

4. **Linear Transformations**

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y+z \\ 2x+y \end{bmatrix}$$

Solution: We can compute that

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) &= \begin{bmatrix} y_1+z_1 \\ 2x_1+y_1 \end{bmatrix} + \begin{bmatrix} y_2+z_2 \\ 2x_2+y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1+y_2+z_1+z_2 \\ 2(x_1+x_2)+y_1+y_2 \end{bmatrix} = L\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) \end{aligned}$$

and

$$cL\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} y+z \\ 2x+y \end{bmatrix} = \begin{bmatrix} cy+cz \\ 2cx+cy \end{bmatrix} = L\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right).$$

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why? **Solution:** Row reducing this matrix gives $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

Each row is non-zero, so there is a solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^2$, and thus the function is onto. But the columns aren't linearly independent, or equivalently the nullspace is non-trivial, so the function is not one-to-one. Or alternatively again, we can observe that $L(1, -2, 2) = (0, 0)$.

3. **Linear Independence**

(a) *Using the formal definition of linear independence*, show that the set

$$\left\{ \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is linearly independent.

Solution: Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. We can set this up as

$$a \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2a - b + c &= 0 \\ a + 3b &= 0 \\ b + 2c &= 0. \end{aligned}$$

This is a homogeneous system of linear equations, so we can set up a matrix row reduction

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so we see the only solution is $a = b = c = 0$. Thus by definition, the vectors are linearly independent.

(b) Determine whether each of the following sets is linearly dependent or linearly independent.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$$

Solution: For the first set, we can write down the matrix

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 5 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

So this set is linearly dependent. Alternatively, we could notice that

$$4 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}.$$

For the second, we write the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & 1 & 2 \\ 2 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus this set is linearly independent.

2. **Vector Equations and Spans**

(a) Is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$?

Solution: No. We set up the equation

$$x \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

which gives the matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is inconsistent. So $\mathbf{v}$ is not in the span of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(b) Write the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as a linear combination of the vectors

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$

Solution: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$

1. **Systems of Linear Equations**

Use row reduction to find all solutions to each system of equations.

(a)

$$\begin{aligned} 2x + y + 3z &= 1 \\ 4x + 3y + 5z &= 1 \\ 6x + 5y + 5z &= -3 \end{aligned}$$

Solution: $x = -3, y = 1, z = 2$, or

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(b)

$$\begin{aligned} 2x + 3y + z &= 1 \\ x + y + z &= 3 \\ 3x + 4y + 2z &= 4 \end{aligned}$$

Solution: $x = 8 - 2z, y = -5 + z$, or

$$\left\{ \begin{bmatrix} 8 \\ -5 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$