Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 4
Due Midnight on Thursday, October 1

This week's mastery quiz has six topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 8 and 7. You may answer *one* of the other six topics if you did not get a "mastery" grade on it already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

8. Basis and Dimension
7. Subspaces
6. Matrix Inverses
5. Matrix Multiplication
4. Linear Transformations
3. Linear Independence
2. Vector Equations and Spanning
1. Systems of Linear Equations

8. **Basis and Dimension** Let $A = \begin{bmatrix} 1 & 3 & 2 & 2 \\ -1 & 1 & 2 & 2 \\ 3 & 9 & 5 & 1 \end{bmatrix}$

(a) Find a basis for the nullspace of A.

(b) Find a basis for the columnspace of A.

(c) Find a basis for the rowspace of A.

(d) What is the rank of A?

Solution: We row reduce to get $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$. Then

(a) A basis for the nullspace is $\{(-4, 4, -5, 1)\}$.

(b) A basis for the columnspace is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \right\}.$$

(c) A basis for the rowspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \right\}.$$

(d) The rank is 3.

7. **Subspaces** For each of the following sets, determine whether it is a subspace of $\mathbb{R}^3$, and justify your answer using the definition of subspace.

(a) $U = \{(x, y, z) : x + y = 1\}$

(b) $V = \{(x, y, z) : x + y = 3z\}$

(c) $W = \{(x, y, z) : x^2 = y^2\}$.

Solution:

(a) The vector $(1, 0, 0)$ is in U, but $2 \cdot (1, 0, 0) = (2, 0, 0)$ is not in U. So U is not a subspace.

(b) V is a subspace. We need to check three things.

i. $0 + 0 = 3 \cdot 0$ so $\mathbf{0} \in V$.

ii. Suppose $(x, y, z), (a, b, c) \in V$. Then $x + y = 3z$ and $a + b = 3c$, so $(x + a) + (y + b) = 3(z + c)$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$$

is in V.

iii. Suppose $(x, y, z) \in V$, and $r \in \mathbb{R}$. Then $x + y = 3z$ so $rx + ry = 3rz$, and so

$$r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$$

is in V.

Thus by definition, V is a subspace of $\mathbb{R}^3$.

(c) The vectors $(1, 1, 0)$ and $(-1, 1, 0)$ are both in W, but $(1, 1, 0)+(-1, 1, 0) = (0, 2, 0)$ is not in W, so W is not a subspace.

6. **Matrix Inverses**

(a) Find the inverse of $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -2 & -3 \\ 1 & 3 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 2 & -2 & -3 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 0 & 5 & 3 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -3 & -2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -3 & -2 & 1 \\ 0 & 0 & -1 & 8 & 5 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -7 & -4 & 2 \\ 0 & 1 & 0 & 5 & 3 & -1 \\ 0 & 0 & 1 & -8 & -5 & 2 \end{bmatrix}$$

so $A^{-1} = \begin{bmatrix} -7 & -4 & 2 \\ 5 & 3 & -1 \\ -8 & -5 & 2 \end{bmatrix}$.

(b) Which of the following matrices are invertible? Justify your answers.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution: The first isn't invertible becasue the rows are linearly dependent. The second is invertible because the rows are linearly independent. The third isn't invertible since it isn't square. The fourth isn't invertile since the columns are linearly dependent.

5. **Matrix Multiplication**

Compute, or write that the computation is undefined:

(a) $\begin{bmatrix} 5 & 7 & 3 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 5 & 1 \end{bmatrix} =$

Solution: $\begin{bmatrix} 34 & 53 \\ 19 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & -1 \end{bmatrix} =$ **Solution:** $\begin{bmatrix} 6 & 26 \\ 1 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ -1 & 3 & 3 \\ 2 & -5 & 1 \end{bmatrix} =$

Solution: Undefined

(d) $\begin{bmatrix} 1 & 4 & 2 \\ -1 & 3 & 3 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} =$

Solution: $\begin{bmatrix} -4 \\ -7 \\ 25 \end{bmatrix}$

4. **Linear Transformations**

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x - z \\ x + 2y \\ 2x - 2y - z \end{bmatrix}$$

Solution: We can compute that

$$L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - z_1 \\ x_1 + 2y_1 \\ 2x_1 - 2y_1 - z_1 \end{bmatrix} + \begin{bmatrix} 3x_2 - z_2 \\ x_2 + 2y_2 \\ 2x_2 - 2y_2 - z_2 \end{bmatrix}$$
$$= \begin{bmatrix} 3(x_1 + x_2) - (z_1 + z_2) \\ (x_1 + x_2) + 2(y_1 + y_2) \\ 2(x_1 + x_2) - 2(y_1 + y_2) - (z_1 + z_2) \end{bmatrix} = L\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right)$$

and

$$cL\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c\begin{bmatrix} 3x - z \\ x + 2y \\ 2x - 2y - z \end{bmatrix} = \begin{bmatrix} 3cx - cz \\ cx + 2cy \\ 2cx - 2cy - cz \end{bmatrix} = L\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right).$$

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 0 \\ 2 & -2 & -1 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why? **Solution:** Row reducing this matrix gives $\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{bmatrix}$.

Because there is a non-zero row, there are $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions, and thus the function is not onto. Because there is a free variable, the nullspace is nontrivial, and thus the function is not one-to-one.

3. **Linear Independence**

(a) *Using the formal definition of linear independence*, show that the set

$$\left\{ \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

is linearly independent.

Solution: Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. We can set this up as

$$a \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} a + 3b &= 0 \\ 2b + 3c &= 0 \\ -2a - 2b + c &= 0 \end{aligned}$$

This is a homogeneous system of linear equations, so we can set up a matrix row reduction

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ -2 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so we see the only solution is $a = b = c = 0$. Thus by definition, the vectors are linearly independent.

(b) Find a linear dependence relationship among the following vectors (or, equivalently, write one as a linear combination of the others):

$$\left\{ \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \right\}$$

Solution:

We can write down the matrix

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 3 & 3 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus if we have $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = 0$ then we must have $b = c$ and $2a - 3c = 0$. So our linear independence relation is $3a - 2b - 2c = 0$. (Or, alternately, we can write something like $a = \frac{2}{3}b + \frac{2}{3}c$.)

2. **Vector Equations and Spans**

(a) Is the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$?

Solution: No. We set up the equation

$$x \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

which gives the matrix

$$\begin{bmatrix} 1 & 4 & 0 \\ -1 & 0 & 3 \\ 2 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 3 \\ 0 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is inconsistent. So $\mathbf{v}$ is not in the span of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(b) Write down and solve an explicit vector equation for writing the vector $\mathbf{b} = \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$.

Solution: We want to solve the equation $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{b}$, or

$$x \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} + y \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}.$$

Then we get

$$\begin{bmatrix} 3 & 5 & -1 \\ 2 & 1 & -3 \\ -2 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & -7 & -7 \\ 0 & 9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we can write $\mathbf{b} = -2\mathbf{v}_1 + \mathbf{v}_2$.

1. **Systems of Linear Equations**

   Use row reduction to find all solutions to each system of equations.

(a)

$$\begin{aligned} x + 3y - 2z &= 3 \\ 2x - y + z &= 4 \\ -x + 4y + 3z &= 5 \end{aligned}$$

Solution: $x = 2, y = 1, z = 1$, or

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b)

$$\begin{aligned} 4x + 2y - z &= 4 \\ x - y + 2z &= 6 \\ 6x + 3z &= 5 \end{aligned}$$

Solution: This system is inconsistent. If we row reduce we get

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which third equation is $0 = 1$. So the set of solutions is empty, $\varnothing$ or $\{\}$.