

Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 5
Due Midnight on Thursday, October 8

This week's mastery quiz has ten topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 10 and 9. You may answer *one* of the other six topics if you did not get a "mastery" grade on it already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

10. Vector Space Linear Transformations
9. Vector Spaces and Subspaces
8. Basis and Dimension
7. Subspaces
6. Matrix Inverses
5. Matrix Multiplication
4. Linear Transformations
3. Linear Independence
2. Vector Equations and Spanning
1. Systems of Linear Equations

10. Vector Space Linear Transformations

- (a) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of real-valued functions, and let $U = \{f(x) : f(1) = f(0)f(2)\}$. Is U a subspace of V ? Prove your answer.

Solution: Let $f(x) = 1$ be the constant function. Then $f(1) = 1 = f(0)f(2)$, so $f \in U$. But $2f(1) = 2$ and $2f(0)2f(2) = 4$, so $2f \notin U$. Thus U is not closed under scalar multiplication, and so isn't a subspace.

- (b) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of real-valued functions, and let $W = \{f : f(1) = f(0) + f(2)\}$. Is W a subspace of V ? Prove your answer.

Solution:

i. $\mathbf{0} \in W$ because $0 = 0 + 0$.

ii. If $f, g \in W$, then

$$(f+g)(1) = f(1) + g(1) = f(0) + f(2) + g(0) + g(2) = (f+g)(0) + (f+g)(2)$$

so $f+g \in W$ and W is closed under addition.

iii. if $f \in W, r \in \mathbb{R}$, then

$$(rf)(1) = rf(1) = r(f(0) + f(2)) = (rf)(0) + (rf)(2)$$

so $rf \in W$ and W is closed under scalar multiplication.

Thus W is a subspace of V .

9. Vector Spaces and Subspaces

- (a) Let $L : \mathcal{P}_3(x) \rightarrow \mathcal{P}_2(x)$ be defined by $L(f(x)) = f'(x) - xf''(x)$. Is L a linear function? Prove your answer.

Solution:

i.

$$L(f+g) = (f+g)'(x) - x(f+g)''(x) = f'(x) + x f''(x) + g'(x) + x g''(x) = L(f(x)) + L(g(x))$$

ii.

$$L(rf(x)) = (rf)'(x) - x(rf)''(x) = rf'(x) - rxf''(x) = r(f'(x) - xf''(x)) = rL(f(x)).$$

Thus L is linear by definition.

- (b) Let $T : \mathcal{P}_2(x) \rightarrow \mathcal{P}_4(x)$ be defined by $T(f(x)) = (f(x))^2$. Is T a linear function? Prove your answer.

Solution: Let $f(x) = x$. Then $T(2f) = (2x)^2 = 4x^2$, but $2T(f) = 2x^2$. So $T(2f) \neq 2T(f)$ and so T is not linear.

8. **Basis and Dimension** Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & 7 \\ 1 & 3 & 0 \\ 0 & 3 & 3 \end{bmatrix}$

- (a) Find a basis for the nullspace of A .
- (b) Find a basis for the columnspace of A .
- (c) Find a basis for the row space of A .
- (d) What is the rank of A ?

Solution: We row reduce to get $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then

- (a) A basis for the nullspace is $\{(-3, -1, 1)\}$.
- (b) A basis for the columnspace is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}.$$

- (c) A basis for the row space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- (d) The rank is 2.

7. **Subspaces** For each of the following sets, determine whether it is a subspace of \mathbb{R}^3 , and justify your answer using the definition of subspace.

- (a) $U = \{(x, y, z) : 2x + 4y - 2z = 0\}$
- (b) $V = \{(x, y, z) : xz = yz\}$
- (c) $W = \{(x, y, z) : z = 2\}$.

Solution:

- (a) U is a subspace. We need to check three things.

i. $2 \cdot 0 + 4 \cdot 0 - 2 \cdot 0 = 0$ so $\mathbf{0} \in V$.

ii. Suppose $(x, y, z), (a, b, c) \in V$. Then $2x + 4y - 2z = 0$ and $2a + 4b - 2c = 0$, so $2(x + a) + 4(y + b) - 2(z + c) = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$$

is in V .

iii. Suppose $(x, y, z) \in V$, and $r \in \mathbb{R}$. Then $2x + 4y - 2z = 0$ so $2rx + 4ry - 2rz = 0$, and so

$$r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$$

is in V .

Thus by definition, V is a subspace of \mathbb{R}^3 .

- (b) The vectors $(1, 1, 1)$ and $(1, 3, 0)$ are both in V , but $(1, 1, 1) + (1, 3, 0) = (2, 4, 1)$ is not. So V is not closed under addition and is not a subspace of \mathbb{R}^3 .
- (c) The vector $(0, 0, 2)$ is in W but $2(0, 0, 2) = (0, 0, 4)$ is not. So W is not closed under scalar multiplication and is not a subspace.

6. Matrix Inverses

- (a) Find the inverse of $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix}$.

Solution: $A^{-1} = \begin{bmatrix} 1 & -1 & -3 \\ 2 & -4 & -9 \\ -1 & 2 & 5 \end{bmatrix}$.

- (b) Which of the following matrices are invertible? Justify your answers.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad [1 \ 3 \ 5 \ 2 \ 4] \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 7 \end{bmatrix}$$

Solution: The first isn't invertible because the rows are linearly dependent. The second is invertible because the rows are linearly independent. The third isn't invertible since it isn't square. The fourth is invertible since we can row-reduce it and get the identity; it's really enough to see that it's upper-triangular, though we haven't talked about that.

5. Matrix Multiplication

Compute, or write that the computation is undefined:

(a) $[1 \ 2 \ 5] \begin{bmatrix} 1 & 3 \\ 5 & -2 \\ 3 & 0 \end{bmatrix} =$

Solution: $[26 \ -1]$

(b) $\begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 3 \\ 2 & 4 & 0 \end{bmatrix} =$

Solution: undefined

(c) $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & -2 \\ 5 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 4 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} =$

Solution: $\begin{bmatrix} 23 & 0 & 14 \\ 7 & 6 & -4 \\ 10 & 11 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 5 & 1 & -1 \end{bmatrix} =$

Solution: $\begin{bmatrix} 22 & 6 & 6 \\ -11 & -1 & 10 \end{bmatrix}$

4. Linear Transformations

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 3y + z \\ 4x - y \end{bmatrix}$$

Solution: We can compute that

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) &= \begin{bmatrix} x_1 - 3y_1 + z_1 \\ 4x_1 - y_1 \end{bmatrix} + \begin{bmatrix} x_2 - 3y_2 + z_2 \\ 4x_2 - y_2 \end{bmatrix} \\ &= \begin{bmatrix} (x_1 + x_2) - 3(y_1 + y_2) + (z_1 + z_2) \\ 4(x_1 + x_2) - (y_1 + y_2) \end{bmatrix} = L\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right) \end{aligned}$$

and

$$cL\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = c \begin{bmatrix} x - 3y + z \\ 4x - y \end{bmatrix} = \begin{bmatrix} cx - 3cy + cz \\ 4cx - cy \end{bmatrix} = L\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right).$$

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 1 & -3 & 1 \\ 4 & -1 & 0 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why? **Solution:** Row reducing this matrix gives $\begin{bmatrix} 1 & -3 & 1 \\ 0 & 11 & -4 \end{bmatrix}$.

Because there is no non-zero row, there is a solution to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^2$, and thus the function is onto. Because there is a free variable, the nullspace is nontrivial, and thus the function is not one-to-one.

3. Linear Independence

(a) Using the formal definition of linear independence, show that the set

$$\left\{ \mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is linearly independent.

Solution: Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. We can set this up as

$$\begin{aligned} a \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ -b + 2c &= 0 \\ a + b + c &= 0 \\ 4a + 2b + c &= 0 \end{aligned}$$

This is a homogeneous system of linear equations, so we can set up a matrix row reduction

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so we see the only solution is $a = b = c = 0$. Thus by definition, the vectors are linearly independent.

- (b) Find a linear dependence relationship among the following vectors (or, equivalently, write one as a linear combination of the others):

$$\left\{ \mathbf{x} = \begin{bmatrix} 5 \\ 13 \\ 2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Solution:

We can write down the matrix

$$\begin{bmatrix} 5 & 4 & 1 \\ 13 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/7 \\ 0 & 1 & 3/7 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus if we have $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ then we must have $a = c/7$ and $b = -3c/7$. So we have $a\mathbf{x} - 3\mathbf{y} + 7\mathbf{z} = \mathbf{0}$.

2. Vector Equations and Spans

- (a) Is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$?

Solution: No. We set up the equation

$$x \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which gives the matrix

$$\begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is inconsistent. So \mathbf{v} is not in the span of \mathbf{u}_1 and \mathbf{u}_2 .

- (b) Write down and solve an explicit vector equation for writing the vector $\mathbf{b} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

as a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

Solution: We want to solve the equation $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{b}$, or

$$x \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}.$$

Then we get

$$\begin{bmatrix} 1 & 2 & 4 \\ -3 & 1 & 9 \\ 5 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we can write $\mathbf{b} = -2\mathbf{v}_1 + 3\mathbf{v}_2$.

1. Systems of Linear Equations

Use row reduction to find all solutions to each system of equations.

(a)

$$\begin{aligned} 4x + 2y - z &= 4 \\ x - y + 2z &= 6 \\ 6x + 3z &= 5 \end{aligned}$$

Solution: This system is inconsistent. If we row reduce we get

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which third equation is $0 = 1$. So the set of solutions is empty, \emptyset or $\{\}$.

(b)

$$\begin{aligned} 2x - y + 3z &= 3 \\ -x + 2y - 2z &= 1 \\ x + 4y &= 9 \end{aligned}$$

Solution: $x = 7/3 - 4z/3, y = 5/3 + z/3$, or

$$\left\{ \begin{bmatrix} 7/3 - 4z/3 \\ 5/3 + z/3 \\ z \end{bmatrix} \right\}$$