Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 6
Due Midnight on Thursday, October 15

This week's mastery quiz has eight topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 12 and 11. You may answer *one* of the other topics if you did not get a "mastery" grade on it already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

12. Matrices of Linear Transformations
11. Bases and Coordinates
10. Vector Space Linear Transformations
9. Vector Spaces and Subspaces
8. Basis and Dimension
7. Subspaces
6. Matrix Inverses
4. Linear Transformations

12. **Matrices of Linear Transformations**

(a) Let $L : \mathcal{P}_3(x) \to \mathcal{P}_2(x)$ be defined by $L(f) = 3f'(x) - f'''(x)$. Give a matrix for L with respect to the standard bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$.
Solution:

$$\begin{bmatrix} 0 & 3 & 0 & -6 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y - z \\ 2x + 3y + 2z \\ x - 3y + z \end{bmatrix}.$$

Give a matrix for T with respect to the basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Solution:
We have

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & 2 & 4 \\ -1 & 0 & -3 \\ -1 & 2 & 1 \end{bmatrix}$$

11. **Bases and Coordinates**

(a) Prove that $\{1 + x, 3 - x^2, 4x + 2x^2\}$ is a basis for $\mathcal{P}_2(x)$.
Solution: Suppose $a(1 + x) + b(3 - x^2) + c(4x + 2x^2) = 0$. Then we get the system of equations

$$\begin{aligned} a + 3b &= 0 \\ a + 4c &= 0 \\ -b + 2c &= 0 \end{aligned}$$

which gives the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 4 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus $a = b = c = 0$. So by definition of linear independence, the three vectors are independent. Since $\mathcal{P}_2(x)$ has dimension 3, they must also span and thus be a basis.

If you want to check spanning directly, you would instead observe that if we want to solve $a(1 + x) + b(3 - x^2) + c(4x + 2x^2) = \alpha + \beta x + \gamma$ then there will always be a solution, since the unaugmented matrix reduces to the identity.

(b) Let $B = \left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}_B$.

Solution:

We want to express the vector $(2, 2, 2)$ as a linear combination of vectors in B. So we want to solve

$$a \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, +c \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 4 & -2 & 3 & 2 \\ 2 & 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 14/19 \\ 0 & 1 & 0 & 12/19 \\ 0 & 0 & 1 & 2/19 \end{array}\right]$$

so the coordinates are

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 14/19 \\ 12/19 \\ 2/19 \end{bmatrix}.$$

10. **Vector Space Linear Transformations**

(a) Let $L : S \to S$ be defined by $L(\{y_k\}) = \{y_k + k\}$, where S is the space of signals. Is L a linear function? Prove your answer.

Solution:

$$L(\{x_k\} + \{y_k\}) = \{x_k + y_k + k\}$$
$$L(\{x_k\}) + L(\{y_k\}) = \{x_k + k\} + \{y_k + k\} == \{x_k + y_k + 2k\}$$

which aren't the same. So L is not a linear transformation.

(b) Let $T : \mathbb{R}^3 \to \mathcal{P}_4(x)$ be defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = ax^4 + (b - c)x^3 + (a + b + c)x.$$

is T a linear function? Prove your answer.

Solution:

i.

$$T\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$$

9. **Vector Spaces and Subspaces**

(a) Let $V = M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$ be the space of 2×2 matrices, and let $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+b+c+d = 1 \right.$
Is U a subspace of V? Prove your answer.

Solution: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A \in U$, but $2A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin U$. So U is not closed under scalar multiplication and is not a subspace.

(b) Let $V = M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$ be the space of 2×2 matrices, and let $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = b + c \right\}$.
Is W a subspace of V? Prove your answer.

Solution:

i. $\mathbf{0} \in W$ because $0 = 0 + 0$.

ii. If $A, B \in W$, then $a_1 = b_1 + c_1$ and $a_2 = b_2 + c_2$, so $a_1 + a_2 = b_1 + b_2 + c_1 + c_2$ and thus $A + B \in W$ and W is closed under addition.

iii. If $A \in W, r \in \mathbb{R}$, then $a = b + c$ so $ra = rb + rc$, so $rA \in W$ and W is closed under scalar multiplication.

Thus W is a subspace of V.

8. **Basis and Dimension** Let $A = \begin{bmatrix} 0 & -2 & 1 & 3 \\ 1 & -1 & 2 & 0 \\ 3 & 1 & 1 & 3 \end{bmatrix}$

(a) Find a basis for the nullspace of A.

(b) Find a basis for the columnspace of A.

(c) Find a basis for the rowspace of A.

(d) What is the rank of A?

Solution: We row reduce to get $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -3 \end{bmatrix}$. Then

(a) A basis for the nullspace is $\{(-3, 3, 3, 1)\}$.

(b) A basis for the columnspace is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(c) A basis for the rowspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}.$$

(d) The rank is 3.

7. **Subspaces** For each of the following sets, determine whether it is a subspace of $\mathbb{R}^3$, and justify your answer using the definition of subspace.

(a) $U = \{(x, y, z) : y = 2\}$

(b) $V = \{(x, y, z) : xy = 0\}$

(c) $W = \{(x, y, z) : 3x = 5z\}$.

Solution:

(a) U is not a subspace, becuase it doesn't contain the zero vector.

(b) V is not a subspace. It contains the zero vector and it's closed under multiplication. But $(0, 1, 0)$ and $(1, 0, 0) \in V$ and $(1, 1, 0) \notin V$ so V isn't closed under addition.

(c) W is a subspace. We need to check three things.

i. $2 \cdot 0 = 5 \cdot 0$ so $\mathbf{0} \in W$.

ii. Suppose $(x, y, z), (a, b, c) \in W$. Then $3x = 5z$ and $3a = 5c$, so $3(x + a) = 5(z + c)$ and thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}$$

is in V.

iii. Suppose $(x, y, z) \in W$, and $r \in \mathbb{R}$. Then $3x+ = 5z$ so $3rx = 5rz$, and so

$$r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$$

is in W.

Thus by definition, W is a subspace of $\mathbb{R}^3$.

6. **Matrix Inverses**

(a) Find the inverse of $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix}$.

Solution: $A^{-1} = \begin{bmatrix} -7 & 8 & -6 \\ 1 & -1 & 1 \\ 3 & -3 & 2 \end{bmatrix}$.

(b) Using part (a), solve the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $A\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Solution:

$$\mathbf{x} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{y} = A^{-1} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}.$$

4. **Linear Transformations**

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 2x+y \\ x+2y+3z \end{bmatrix}$$

Solution:

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why? **Solution:** Row reducing this matrix gives $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Because there is a non-zero row, there is not always a solution to $A\mathbf{x} = \mathbf{b}$, and thus the function is not onto. Because there is a free variable, the nullspace is nontrivial, and thus the function is not one-to-one.