Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 8
Due Midnight on Thursday, October 29

This week's mastery quiz has eleven topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 14 and 13. You may answer *one* of the other topics if you did not get a "mastery" grade on it already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

14. Characteristic Polynomials and Finding Eigensystems
13. Eigenvectors and Determinants
12. Matrices of Linear Transformations
11. Bases and Coordinates
10. Vector Space Linear Transformations
9. Vector Spaces and Subspaces
8. Basis and Dimension
7. Subspaces
4. Linear Transformations
2. Vector Equations and Spanning
1. Systems of Linear Equations

14. **Characteristic Polynomials and Finding Eigensystems**

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find the characteristic polynomial, the eigenvalues, and a basis for each associated eigenspace.

13. **Eigenvectors and Determinants**

(a) Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. Show that $(-1, 0, 1)$ is an eigenvector of A. What is the corresponding eigenvalue?

(b) Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix}$

12. **Matrices of Linear Transformations**

(a) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L(x, y) = (x, y, x + y)$. Give a matrix for L with respect to the bases to $E = \{(1, 1), (1, -1)$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

(b) Let $T : \mathcal{P}_2(x) \to \mathcal{P}_3(x)$ be defined by $T(f(x)) = \int_0^x f(t)\, dt$. Give a matrix for T with respect to the bases $E = \{1, x, x^2\}$ and $F = \{1, x, x^2, x^3\}$.

11. **Bases and Coordinates**

(a) Prove that $\{1+2x, x+2x^2, x^2+2x^3, 2+x^3\}$ is a basis for $\mathcal{P}_3(x)$.

(b) Let $B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find $\begin{bmatrix} 8 \\ 1 \\ 11 \end{bmatrix}_B$.

10. **Vector Space Linear Transformations**

(a) Let $L : \mathcal{P}_2(x) \to \mathbb{R}^2$ be defined by $L(f) = \begin{bmatrix} f(1) - 1 \\ f(3) - 3 \end{bmatrix}$. Is L a linear function? Prove your answer.

(b) Let $T : \mathbb{R}^2 \to \mathcal{F}(\mathbb{R}, \mathbb{R})$ the space of real-valued functions be defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a\sin(x) + b\cos(x).$$

is T a linear function? Prove your answer.

9. **Vector Spaces and Subspaces**

(a) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of real-valued functions, and $U = \{f : f(1) = 3f(4)$. Is U a subspace of V? Prove your answer.

(b) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of real-valued functions, and $U = \{f : f(1) = 4$. Is U a subspace of V? Prove your answer.

8. **Basis and Dimension** Let $A = \begin{bmatrix} -2 & 4 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$.

(a) Find a basis for the nullspace of A.

(b) Find a basis for the columnspace of A.

(c) Find a basis for the rowspace of A.

(d) What is the rank of A?

7. **Subspaces** For each of the following sets, determine whether it is a subspace of $\mathbb{R}^3$, and justify your answer using the definition of subspace.

   (a) $U = \{(x, y, z) : x + 3y = z\}$
   (b) $V = \{(x, y, z) : x + y + z = 3\}$
   (c) $W = \{(x, y, z) : x^2 = 1\}$.

4. **Linear Transformations**

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+y-z \\ 3x \\ 4y-3z \end{bmatrix}$$

(b) Write a matrix for the linear function.

(c) Is this function one-to-one, onto, both, or neither? Why?

2. **Vector Equations and Spans**

(a) Is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$?

(b) Write down and solve an explicit vector equation for writing the vector $\mathbf{b} = \begin{bmatrix} -2 \\ 6 \\ -6 \end{bmatrix}$ as a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

1. **Systems of Linear Equations**

Use row reduction to find all solutions to each system of equations.

(a)

$$\begin{aligned} x - 4y + 2z &= 2 \\ -x + 3y + z &= 4 \\ 2x - y + z &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ x_1 - 5x_2 + x_4 &= 5 \end{aligned}$$