Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 8
Due Midnight on Thursday, October 29

This week's mastery quiz has eleven topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 14 and 13. You may answer *one* of the other topics if you did not get a "mastery" grade on it already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

14. Characteristic Polynomials and Finding Eigensystems
13. Eigenvectors and Determinants
12. Matrices of Linear Transformations
11. Bases and Coordinates
10. Vector Space Linear Transformations
9. Vector Spaces and Subspaces
8. Basis and Dimension
7. Subspaces
4. Linear Transformations
2. Vector Equations and Spanning
1. Systems of Linear Equations

14. **Characteristic Polynomials and Finding Eigensystems**

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find the characteristic polynomial, the eigenvalues, and a basis for each associated eigenspace.

Solution:

$$\begin{aligned} \chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \\ &= (1-\lambda)^2(-\lambda) + 0 + 0 - ((1-\lambda) + (1-\lambda) + 0) \\ &= -\lambda + 2\lambda^2 - \lambda^3 - 2 + 2\lambda = -2 + \lambda + 2\lambda^2 - \lambda^3 \\ &= (\lambda - 2)(\lambda - 1)(1 - \lambda) \end{aligned}$$

has roots $2, 1, -1$. So we need to find an eigenvector for each eigenvalue.

$$A - 2\lambda = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = \text{span}\,\{(1, 1, 1)\}$$

$$A - \lambda = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_1 = \text{span}\,\{(1, -1, 0)\}$$

$$A + \lambda = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{-1} = \text{span}\{(-1, -1, 2)\}$$

13. **Eigenvectors and Determinants**

(a) Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. Show that $(-1, 0, 1)$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution:

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $(-1, 0, 1)$ is an eigenvector with eigenvalue 2.

(b) Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix}$

Solution:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - (7 \cdot 5 \cdot 3 + 8 \cdot 6 \cdot 1 + 9 \cdot 4 \cdot 2)$$
$$= 45 + 84 + 96 - 105 - 48 - 72 = 225 - 225 = 0.$$

(Alternately, you could notice that twice the second row minus the first row is the third row, so the rows are linearly dependent, so the determinant must be zero.)

$$\det \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix} = (-1)^{3+2} \det \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{bmatrix}$$
$$= (-1)\left(1(-1)^{1+1} \det \begin{bmatrix} 2 & 6 \\ 2 & 1 \end{bmatrix} + 3(-1)^{1+3} \det \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}\right)$$
$$= -(2 - 12 + 3(4-2)) = -(-10+6) = 4.$$

0 and 4.

12. **Matrices of Linear Transformations**

(a) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $L(x, y) = (x, y, x + y)$. Give a matrix for L with respect to the bases to $E = \{(1, 1), (1, -1)$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

Solution:

To find the matrix, we compute

$$L(1, 1) = (1, 1, 2) \to (0, -1, 2)$$
$$L(1, -1) = (1, -1, 0) \to (2, -1, 0)$$

$$A = \begin{bmatrix} 0 & 2 \\ -1 & -1 \\ 2 & 0 \end{bmatrix}.$$

(b) Let $T : \mathcal{P}_2(x) \to \mathcal{P}_3(x)$ be defined by $T(f(x)) = \int_0^x f(t)\,dt$. Give a matrix for T with respect to the bases $E = \{1, x, x^2\}$ and $F = \{1, x, x^2, x^3\}$.

Solution: To find the matrix we compute

$$T(1) = \int_0^x 1\,dt = x \to (0, 1, 0, 0)$$
$$T(x) = \int_0^x t\,dt = x^2/2 \to (0, 0, 1/2, 0)$$
$$T(x^2) = \int_0^x t^2\,dt = x^3/3 \to (0, 0, 0, 1/3)$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

11. **Bases and Coordinates**

(a) Prove that $\{1+2x, x+2x^2, x^2+2x^3, 2+x^3\}$ is a basis for $\mathcal{P}_3(x)$.

Solution: Suppose $a(1+2x)+b(x+2x^2)+c(x^2+2x^3)+d(2+x^3)=0$. Then we get the system of equations

$$\begin{aligned} a+2d&=0\\ 2a+b&=0\\ 2b+c&=0\\ 2c+d&=0 \end{aligned}$$

which gives the matrix

$$\begin{bmatrix} 1&0&0&2\\ 2&1&0&0\\ 0&2&1&0\\ 0&0&2&1 \end{bmatrix} \to \begin{bmatrix} 1&0&0&2\\ 0&1&0&-4\\ 0&0&1&8\\ 0&0&0&-15 \end{bmatrix} \to \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}$$

and thus $a=b=c=d=0$. So by definition of linear independence, the three vectors are independent. Since $\mathcal{P}_4(x)$ has dimension 4, they must also span and thus be a basis.

If you want to check spanning directly, you would instead observe that if we want to solve $a(1+2x)+b(x+2x^2)+c(x^2+2x^3)+d(2+x^3)=\alpha+\beta x+\gamma x^2+\delta x^3$ then there will always be a solution, since the unaugmented matrix reduces to the identity.

(b) Let $B=\left\{\begin{bmatrix}3\\1\\4\end{bmatrix}, \begin{bmatrix}1\\5\\2\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}\right\}$ be a basis for $\mathbb{R}^3$. Find $\begin{bmatrix}8\\1\\11\end{bmatrix}_B$.

Solution:

We want to express the vector $(8,1,11)$ as a linear combination of vectors in B. So we want to solve

$$a\begin{bmatrix}3\\1\\4\end{bmatrix}+b\begin{bmatrix}1\\5\\2\end{bmatrix},+c\begin{bmatrix}0\\3\\1\end{bmatrix}=\begin{bmatrix}8\\1\\11\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3&1&0&8\\ 1&5&3&1\\ 4&2&1&11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1&5&3&1\\ 0&-14&-9&5\\ 0&-18&-11&7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1&5&3&1\\ 0&4&2&-2\\ 0&18&11&-7 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1&5&3&1\\ 0&2&1&-1\\ 0&0&2&2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1&5&3&1\\ 0&2&0&-2\\ 0&0&1&1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1&0&3&6\\ 0&1&0&-1\\ 0&0&1&1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1&0&0&3\\ 0&1&0&-1\\ 0&0&1&1 \end{array}\right]$$

so the coordinates are

$$\begin{bmatrix}8\\1\\11\end{bmatrix}_B=\begin{bmatrix}3\\-1\\1\end{bmatrix}.$$

10. **Vector Space Linear Transformations**

(a) Let $L : \mathcal{P}_2(x) \to \mathbb{R}^2$ be defined by $L(f) = \begin{bmatrix} f(1) - 1 \\ f(3) - 3 \end{bmatrix}$. Is L a linear function? Prove your answer.

Solution: $L(0) = (-1, -3)$ so this is not a linear function.
Alternatively, $L(1) = (0, -2)$ and $L(2) = (1, -1) \neq 2(0, -2)$.

(b) Let $T : \mathbb{R}^2 \to \mathcal{F}(\mathbb{R}, \mathbb{R})$ the space of real-valued functions be defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a\sin(x) + b\cos(x).$$

is T a linear function? Prove your answer.

Solution:

i.

$$\begin{aligned} T\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) &= a_1\sin(x) + b_1\cos(x) + a_2\sin(x) + b_2\cos(x) \\ &= (a_1 + a_2)\sin(x) + (b_1 + b_2)\cos(x) = T\left(\begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}\right) \end{aligned}$$

ii.

$$rT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ra\sin(x) + rb\cos(x) = T\left(\begin{bmatrix} ra \\ rb \end{bmatrix}\right)$$

Thus T is linear by definition.

9. **Vector Spaces and Subspaces**

(a) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of real-valued functions, and $U = \{f : f(1) = 3f(4)$. Is U a subspace of V? Prove your answer.

Solution:

i. If $f(x) = 0$ is the zero vector, then we have $0 = 3 \cdot 0$, so $\mathbf{0} \in U$.
ii. Suppose $f, g \in U$. Then $f(1) = 3f(4)$ and $g(1) = 3g(4)$, and so $(f + g)(1) = 3(f + g)(4)$, so $f + g \in U$.
iii. Suppose $f \in U$ and $r \in \mathbb{R}$. Then $f(1) = 3f(4)$, so $rf(1) = 3rf(4)$ and so $rf \in U$.

Thus U is a subspace of V.

(b) Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of real-valued functions, and $U = \{f : f(1) = 4$. Is U a subspace of V? Prove your answer.

Solution:
The zero vector is not in U, since $0(1) = 0 \neq 4$. Thus U is not a subspace.

8. **Basis and Dimension** Let $A = \begin{bmatrix} -2 & 4 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$.

(a) Find a basis for the nullspace of A.

(b) Find a basis for the columnspace of A.

(c) Find a basis for the rowspace of A.

(d) What is the rank of A?

Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{bmatrix}$, so

- The nullspace is $\{\alpha/6, -\alpha/6, \alpha\}$ and so has basis $\{(1, -1, 6)\}$ or $\{1/6, -1/6, 1\}$.
- The columnspace has basis $\{(-2, -5, 3), (4, 1, 3)\}$.
- The rowspace has basis $\{(1, 0, -1/6), (0, 1, 1/6)\}$.
- The rank is 2.

7. **Subspaces** For each of the following sets, determine whether it is a subspace of $\mathbb{R}^3$, and justify your answer using the definition of subspace.

(a) $U = \{(x, y, z) : x + 3y = z\}$

(b) $V = \{(x, y, z) : x + y + z = 3\}$

(c) $W = \{(x, y, z) : x^2 = 1\}$.

Solution:

(a) U is a subspace. We need to check three things.

i. $0 + 3 \cdot 0 = 0$ so $\mathbf{0} \in U$.

ii. Suppose $(x, y, z), (a, b, c) \in U$. Then $x + 3y = z$ and $a + 3b = c$, so $(x + a) + 3(b + y) = (z + c)$ and thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ z + c \end{bmatrix}$$

is in U.

iii. Suppose $(x, y, z) \in U$, and $r \in \mathbb{R}$. Then $x + 3y = z$ so $rx + 3ry = rz$, and so

$$r \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$$

is in U.

Thus by definition, U is a subspace of $\mathbb{R}^3$.

(b) The zero vector isn't in V because $0 + 0 + 0 \neq 3$, so V isn't a subspace.

(c) The zero vector isn't in W because $0^2 \neq 1$, so W isn't a subspace.

4. **Linear Transformations**

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+y-z \\ 3x \\ 4y-3z \end{bmatrix}$$

Solution:

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & 0 \\ 0 & 4 & -3 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why? **Solution:** Row reducing this matrix gives $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Because there is a non-zero row, there is not always a solution to $A\mathbf{x} = \mathbf{b}$, and thus the function is not onto. Because there is not a free variable, the nullspace is trivial, and thus the function is one-to-one.

2. **Vector Equations and Spans**

(a) Is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the span of the vectors $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$?

Solution: No. We set up the equation

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + b \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5b-a \\ 3a+4b \\ 2a+b \end{bmatrix}$$

which gives the matrix

$$\begin{bmatrix} -1 & 5 & 1 \\ 3 & 4 & 0 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is inconsistent. So $\mathbf{v}$ is not in the span of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(b) Write down and solve an explicit vector equation for writing the vector $\mathbf{b} = \begin{bmatrix} -2 \\ 6 \\ -6 \end{bmatrix}$ as a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Solution: We want to solve the equation $x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{b}$, or

$$x \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ -6 \end{bmatrix}.$$

Then we get

$$\begin{bmatrix} 2 & 3 & -2 \\ 5 & 2 & 6 \\ -2 & 1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we can write $\mathbf{b} = -2\mathbf{v}_1 + 2\mathbf{v}_2$.

1. **Systems of Linear Equations**

Use row reduction to find all solutions to each system of equations.

(a)

$$\begin{aligned} x - 4y + 2z &= 2 \\ -x + 3y + z &= 4 \\ 2x - y + z &= 1 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -4 & 2 & 2 \\ 0 & -1 & 3 & 6 \\ 0 & 7 & -3 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -10 & -22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 18 & 39 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -10 & -22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 1 & 13/6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2/6 \\ 0 & 1 & 0 & 3/6 \\ 0 & 0 & 1 & 13/6 \end{array}\right]$$

so we have $x = -1/3, y = 1/2, z = 13/6$.

(b)

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ x_1 - 5x_2 + x_4 &= 5 \end{aligned}$$

Solution: $\left\{\left(\frac{15}{4} - \frac{5\alpha}{8} - \beta, \frac{-1}{4} - \frac{\alpha}{8}, \alpha, \beta\right)\right\}$