Math 2184 §10 Fall 2020
Linear Algebra I Mastery Quiz 9
Due Midnight on Thursday, November 5

This week's mastery quiz has nine topics. **Do not answer all questions.** Please answer the problems on the new topics, labeled 16 and 15. You may answer *one* of the other topics if you did not get a "mastery" grade on it already. If you are retrying a topic you should complete the entire page.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 30 minutes on this quiz. Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

Topics are:

16. Change of Basis
15. Complex and Generalized Eigenvectors
14. Characteristic Polynomials and Finding Eigensystems
13. Eigenvectors and Determinants
12. Matrices of Linear Transformations
11. Bases and Coordinates
9. Vector Spaces and Subspaces
4. Linear Transformations
3. Linear Independence

16. **Change of Basis**

Let

$$E = \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} \right\}, \qquad F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find the transition matrix from E to F.

Solution:

It's easiest to compute the transition matrices to the standard basis and combine. We have

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 3 & 2 \\ 4 & -1 & 5 \end{bmatrix}$$

as the transition matrix from E to the standard basis. We have

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

as the transition matrix from F to the standard basis. So we need to compute B^{-1}:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -5 & 2 \end{bmatrix}$$

So we have

$$B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ -1 & -5 & 2 \end{bmatrix}$$

and the transition matrix from E to F is

$$B^{-1}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ -1 & -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 1 & 3 & 2 \\ 4 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 8 & -4 \\ 1 & -18 & 3 \end{bmatrix}.$$

15. **Complex and Generalized Eigenvectors**

(a) Let $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$. Find all (complex!) eigenvalues of A and find a (possibly complex) eigenvector for each eigenvalue.

Solution: We have

$$\chi_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 3 \\ -3 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 9 = \lambda^2 - 2\lambda + 10$$

and by the quadratic formula we have

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm \frac{1}{2}\sqrt{-36} = 1 \pm 3i.$$

Set $\lambda = 1 + 3i$, and we have

$$A - \lambda I = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} 3 & 3i \\ 0 & 0 \end{bmatrix}$$

and thus an eigenvector would be $(-i, 1)$. By conjugation, we know that an eigenvector for $\overline{\lambda}$ would be $(i, 1)$.

(b) Let $B = \begin{bmatrix} 1 & -8 & 3 \\ 1 & 4 & -1 \\ 1 & -4 & 3 \end{bmatrix}$. Find an eigenvector of eigenvalue 2, and then find a generalized eigenvector of eigenvalue 2.

Solution:

We have

$$B - 2I = \begin{bmatrix} -1 & -8 & 3 \\ 1 & 2 & -1 \\ 1 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & -6 & 2 \\ 0 & -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$E_2 = \ker(B - 2I) = \{(\alpha, \alpha, 3\alpha)\}.$$

So $(1, 1, 3)$ is an eigenvector of eigenvalue 2.

$$(B - 2I)^2 = \begin{bmatrix} -1 & -8 & 3 \\ 1 & 2 & -1 \\ 1 & -4 & 1 \end{bmatrix}^2 = \begin{bmatrix} -4 & -20 & 8 \\ 0 & 0 & 0 \\ -4 & -20 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has kernel $\{(-5\alpha + 2\beta, \alpha, \beta)\}$. So the easy choices here are $(-5, 1, 0)$ or $(2, 0, 1)$, but there are lots of options. It just can't be in the span of $(1, 1, 3)$.

14. **Characteristic Polynomials and Finding Eigensystems**

Let $A = \begin{bmatrix} -6 & -8 & 6 \\ 0 & 4 & 0 \\ -3 & 6 & 3 \end{bmatrix}$. Find the characteristic polynomial, the eigenvalues, and a basis for each associated eigenspace.

Solution:

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -6-\lambda & -8 & 6 \\ 0 & 4-\lambda & 0 \\ -3 & 6 & 3-\lambda \end{bmatrix}$$
$$= (4-\lambda)((-6-\lambda)(3-\lambda) + 18) = (4-\lambda)(\lambda^2 + 3\lambda) = (4-\lambda)(3+\lambda)\lambda$$

has roots $-3, 0, 4$ So we need to find an eigenvector for each eigenvalue.

$$A + 3I = \begin{bmatrix} -3 & -8 & 6 \\ 0 & 7 & 0 \\ -3 & 6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_{-3} = \text{span}\,\{(2, 0, 1)\}$$

$$A - 0I = \begin{bmatrix} -6 & -8 & 6 \\ 0 & 4 & 0 \\ -3 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_0 = \text{span}\,\{(1, 0, 1)\}$$

$$A - 4I = \begin{bmatrix} -10 & -8 & 6 \\ 0 & 0 & 0 \\ -3 & 6 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & -6 & 1 \\ 0 & -84 & 28 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_4 = \text{span}\{(1, 1, 3)\}$$

13. **Eigenvectors and Determinants**

(a) Let $A = \begin{bmatrix} 3 & 3 & -2 \\ 4 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Show that $(1, 3, 5)$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution:

$$\begin{bmatrix} 3 & 3 & -2 \\ 4 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}.$$

Thus $(1, 3, 5)$ is an eigenvector with eigenvalue 2.

(b) Compute $\det \begin{bmatrix} 4 & 1 & 2 \\ 3 & 1 & -1 \\ 5 & 2 & 2 \end{bmatrix}$ and $\det \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & 6 & 2 \\ 0 & 0 & 4 & 0 \\ 2 & 5 & 1 & 4 \end{bmatrix}$

Solution:

$$\det \begin{bmatrix} 4 & 1 & 2 \\ 3 & 1 & -1 \\ 5 & 2 & 2 \end{bmatrix} = 8 - 5 + 12 - 10 + 8 - 6 = 7.$$

$$\det \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 2 & 6 & 2 \\ 0 & 0 & 4 & 0 \\ 2 & 5 & 1 & 4 \end{bmatrix} = 4(-1)^{3+3} \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 5 & 4 \end{bmatrix}$$

$$= 4(8 + 4 + 10 - 4 - 10 - 8) = 0.$$

Alternately you can observe that the second row is a multiple of the first, and thus the determinant must be 0.

12. **Matrices of Linear Transformations**

(a) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(x, y, z) = (x + 3y, y + 3z, x + y + z)$. Give a matrix for L with respect to the bases to $E = \{(1, 0, 1), (1, 0, -1), (0, 1, 0)$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

Solution:

To find the matrix, we compute

$$\begin{aligned} L(1,0,1) &= (1,3,2) \to (-2,1,2) \\ L(1,0,-1) &= (1,-3,0) \to (4,-3,0) \\ L(0,1,0) &= (3,1,1) \to (2,0,1) \\ A &= \begin{bmatrix} -2 & 4 & 2 \\ 1 & -3 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \end{aligned}$$

(b) Let $T : \mathcal{P}_3(x) \to \mathbb{R}^2$ be defined by $T(f(x)) = (f(1) + f(2), f(4) - f(3))$. Find a matrix for T with respect to the bases $E = \{1, x, x^2\}$ and $F = \{(1, 1), (1, -1)\}$.

Solution: To find the matrix we compute

$$T(1) = (2,0) \to (1,1)T(x) \quad = (3,1) \to (2,1)T(x^2) = (5,7) \to (6,-1)A = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 1 & -1 \end{bmatrix}.$$

11. **Bases and Coordinates**

(a) Prove that $\{1 + 2x + 3x^2, x - 2x^2, 1 + 3x + 2x^2\}$ is a basis for $\mathcal{P}_2(x)$. (Please explicitly use the formal definition of a basis.)

Solution: Suppose $a(1 + 2x + 3x^2) + b(x - 2x^2) + c(1 + 3x + x^2) = 0$. Then we get the system of equations

$$\begin{aligned} a + c &= 0 \\ 2a + b + 3c &= 0 \\ 3a - 2b + 2c &= 0 \end{aligned}$$

which gives the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus $a = b = c = 0$. So by definition of linear independence, the three vectors are independent. Since $\mathcal{P}_2(x)$ has dimension 3, they must also span and thus be a basis.

If you want to check spanning directly, you would instead observe that if we want to solve $a(1 + 2x + 3x^2) + b(x - 2x^2) + c(1 + 3x + x^2) = \alpha + \beta x + \gamma x^2$ then there will always be a solution, since the unaugmented matrix reduces to the identity.

(b) Let $B = \left\{ \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$. Find $\begin{bmatrix} 13 \\ -16 \\ 9 \end{bmatrix}_B$.

Solution:

We want to express the vector $(7, -4, 15)$ as a linear combination of vectors in B. So we want to solve

$$a \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}, +c \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ -16 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 1 & 13 \\ -3 & -5 & 3 & -16 \\ 1 & 1 & 4 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 4 & 9 \\ 0 & -4 & -15 & -23 \\ 0 & -2 & 15 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 4 & 9 \\ 0 & 2 & -15 & -11 \\ 0 & 0 & -45 & -45 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

so the coordinates are

$$\begin{bmatrix} 13 \\ -16 \\ 9 \end{bmatrix}_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

9. **Vector Spaces and Subspaces**

(a) Let $V = \mathcal{P}_2(x)$ be the space of degree-2 polynomials, and $U = \{a_0 + a_1x + a_2x^2 : a_0 + a_1 + a_2 = a_0 + 2a_1 + 4a_2$. Is U a subspace of V? Prove your answer.

Solution:

(b) Let $V = \mathcal{P}_2(x)$ be the space of degree-2 polynomials, and $W = \{a_0 + a_1x + a_2x^2 : a_0 + a_1 + a_2^2 = 0\}$. Is W a subspace of V? Prove your answer.

Solution:

4. **Linear Transformations**

(a) Prove directly from the definition that the following function is linear:

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x + y + z \\ 3x + w \end{bmatrix}$$

Solution:

(b) Write a matrix for the linear function.

Solution: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$

(c) Is this function one-to-one, onto, both, or neither? Why?

Solution: Row reducing this matrix gives $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix}$.

Because there is no non-zero row, there is always a solution to $A\mathbf{x} = \mathbf{b}$, and thus the function is onto. But there are two free variables, so solutions are not unique; in particular, there's a non-trivial kernel. Thus the function is not one-to-one.

3. **Linear Independence**

(a) *Using the formal definition of linear independence*, show that the set

$$\left\{ \mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} \right\}$$

is linearly independent.

Solution: Suppose $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. We can set this up as

$$a \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a + 3b + 2c = 0$$

$$4a - 3c = 0$$

$$2a + b + 2c = 0$$

This is a homogeneous system of linear equations, so we can set up a matrix row reduction

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & -3 \\ 2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so we see the only solution is $a = b = c = 0$. Thus by definition, the vectors are linearly independent.

(b) Find a linear dependence relationship among the following vectors (or, equivalently, write one as a linear combination of the others):

$$\left\{ \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 8 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Solution:

We want to solve $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$. We can write down the matrix

$$\begin{bmatrix} 3 & 2 & 2 \\ 1 & -2 & 2 \\ 4 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus if we have $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = 0$ then we must have $a = -c$ and $2b = c$. So we have $-2\mathbf{x} + \mathbf{y} + 2\mathbf{z} = \mathbf{0}$.