9 Divergence and Differential Forms

Recall that in section 7 we defined the line integral, then looked for a way to avoid it. We saw that if a vector field $\vec{F}$ is "conservative" then line integrals are path independent—we can find some potential function f so that $\nabla f = \vec{F}$, and then evaluate f on the endpoints rather than integrating $\vec{F}$ over the whole curve. We then saw that if a vector field has $\nabla \times \vec{F} = 0$, it is conservative.

Now we want to do the same thing for surface integrals. In section 8 we defined the surface integral, then saw that if our field $\vec{G}$ is a curl field—that is, $\vec{G} = \nabla \times \vec{F}$ for some vector potential $\vec{F}$—then instead of computing the surface integral of $\vec{G}$ over a surface, we can integrate $\vec{F}$ over the boundary. But how can we tell when a field is a curl field?

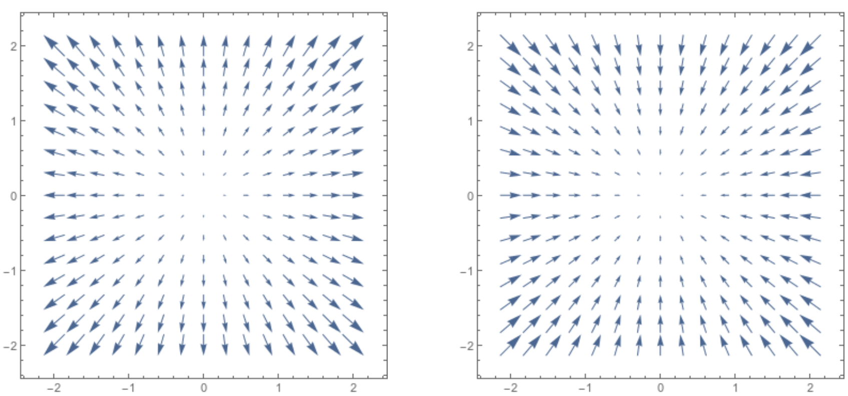
9.1 The divergence of a vector field

Just as we defined the curl to be the density of the circulation, which is the line integral of a vector field, we will define the *divergence* to measure the density of the flux. Thus divergence will measure the extent to which a vector field flows into or out of a region.

Definition 9.1. The *divergence* or *flux density* of a vector field is

$$\nabla \cdot \vec{F}(x, y, z) = \lim_{\text{volume}\to 0} \frac{\int_S \vec{F} \cdot d\vec{A}}{\text{volume of } S}$$

where S is a sphere centered at (x, y, z) oriented outwards.



The divergence at the origin on the left is positive; on the right it is negative.

Proposition 9.2. *We can compute the divergence with*

$$\begin{aligned}\nabla \cdot \vec{F} &= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.\end{aligned}$$

Proof. It doesn't actually matter whether we use a small sphere or a small box. So let's imagine computing the flux density of $\vec{F}$ over a small box oriented outwards, with side lengths $\Delta x, \Delta y, \Delta z$ and the corner closest to the origin at (x_0, y_0, z_0).

Then the box has six faces. Let's consider the top and bottom. The bottom face has vector $\Delta x \Delta y(-\vec{k})$. Since the box is small the vector field is approximately constant at the value $\vec{F}(x_0, y_0, z_0)$, so the integral over the bottom face is approximately

$$\vec{F}(x_0, y_0, z_0) \cdot (-\Delta x \Delta y)\vec{k} = -F_3(x_0, y_0, z_0)\Delta x \Delta y.$$

Looking at the top now, we see that the face has vector $\Delta x \Delta y \vec{k}$, but the vector field is now approximately $\vec{F}(x_0, y_0, z_0 + \Delta z)$. The integral is then approximately

$$\vec{F}(x_0, y_0, z_0 + \Delta z) \cdot \Delta x \Delta y \vec{k} = F_3(x_0, y_0, z_0 + \Delta z)\Delta x \Delta y.$$

Adding these two together gives

$$F_3(x_0, y_0, z_0 0 + \Delta z)\Delta x \Delta y - F_3(x_0, y_0, z_0)\Delta x \Delta y = \frac{F_3(x_0, y_0, z_0 + \Delta z) - F_3(x_0, y_0, z_0)}{\Delta z}\Delta x \Delta y \Delta z.$$

To compute the flux density, we divide by the volume, which is $\Delta x \Delta y \Delta z$. Then taking the limit gives us

$$\lim_{\Delta z \to 0} \frac{F_3(x_0, y_0, z_0 + \Delta z) - F_3(x_0, y_0, z_0)}{\Delta z} = \frac{\partial F_3}{\partial z}.$$

We run through the same calculations for the two faces on the side to get $\frac{\partial F_1}{\partial x}$ and the front and back to get $\frac{\partial F_2}{\partial y}$. Adding all three components together gives us

$$\nabla \cdot \vec{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

as desired.

□

Example 9.3. Let's compute the divergence of $\vec{F}(\vec{r}) = \vec{r}$ at the origin.

Even without doing any computations, we can see that the divergence must be positive, since the flux is definitely outwards.

First, let's use the geometric definition. A sphere of radius a has outward flux of $4\pi a^3$. We can see this by arguing that the vector field is always perpendicular to the sphere, so we have $\int_S \vec{F} \cdot d\vec{dA}$ is equal to the magnitude of $\vec{F}$ times the surface area of the sphere, which is $a \cdot 4\pi a^2 = 4\pi a^3$. Alternatively, we have

$$\text{Flux} = \int_S \vec{F} \cdot \frac{\vec{r}}{\|\vec{r}\|}\, dA = \int_S \|\vec{r}\|\, dA = a \int_S 1\, da = a \cdot 4\pi a^2 = 4\pi a^3.$$

The volume is $4/3\pi a^3$, so the flux density is

$$\lim_{a\to 0} \frac{4\pi a^3}{4/3\pi a^3} = \lim_{a\to 0} 3 = 3.$$

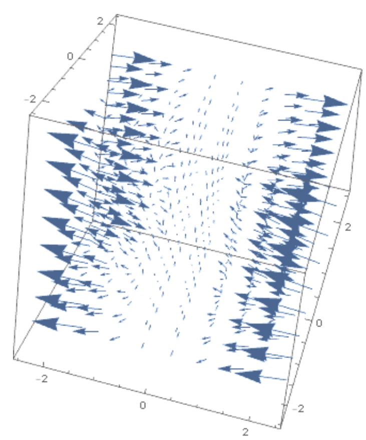
Alternately, we can compute with the algebraic definition. We have

$$\nabla \cdot \vec{F} = \nabla \cdot (x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

As usual, you can see that we'd much prefer to compute the divergence using the algebraic definition, rather than the geometric definition.

Example 9.4. Let $\vec{F}(x, y, z) = x^2y\vec{i} + \cos(z)\vec{j} + \sin(z)\vec{k}$. Then

$$\nabla \cdot \vec{F}(x, y, z) = 2xy + 0 + \cos(z).$$



Let $\vec{G}(x, y) = -y\vec{i} + x\vec{j}$. Then $\nabla \cdot \vec{G} = 0 + 0 = 0$. $\vec{G}$ is rotating, but it has zero divergence since there's no net flux into or out of any region.

Any constant vector field has zero divergence since the exact same amount of fluid is entering and leaving every point.

Definition 9.5. We say that $\vec{F}$ is *divergence free* or *solenoidal* or *incompressible* if $\nabla \cdot \vec{F} = 0$ whenever $\vec{F}$ is defined.

Physically, this means that the density of fluid is conserved—on net it isn't flowing into or out of any region.

Example 9.6. Let $\vec{E} = \frac{\vec{r}}{\|\vec{r}\|^p}$. Let's find the divergence, and determine for what p this field is solenoidal.

We have

$$\begin{aligned}
\frac{\partial E_1}{\partial x} &= \frac{\partial}{\partial x} \frac{x}{(x^2+y^2+z^2)^{p/2}} \\
&= \frac{(x^2+y^2+z^2)^{p/2} - x\frac{p}{2}(x^2+y^2+z^2)^{p/2-1}(2x)}{(x^2+y^2+z^2)^p} \\
&= \frac{(x^2+y^2+z^2)^{p/2} - px^2(x^2+y^2+z^2)^{p/2-1}}{(x^2+y^2+z^2)^p} \\
&= \frac{(x^2+y^2+z^2) - px^2}{(x^2+y^2+z^2)^{p/2+1}} \\
\frac{\partial E_2}{\partial y} &= \frac{(x^2+y^2+z^2) - py^2}{(x^2+y^2+z^2)^{p/2+1}} \\
\frac{\partial E_3}{\partial z} &= \frac{(x^2+y^2+z^2) - pz^2}{(x^2+y^2+z^2)^{p/2+1}} \\
\nabla \cdot \vec{E} &= \frac{3(x^2+y^2+z^2) - px^2 - py^2 - pz^2}{(x^2+y^2+z^2)^{p/2+1}} \\
&= \frac{3-p}{(x^2+y^2+z^2)^{p/2}} = \frac{3-p}{\|\vec{r}\|^p}.
\end{aligned}$$

Thus $\vec{E}$ is solenoidal if and only if $p = 3$. (Recall that in this case, we have an inverse-square law, as appears in equations for electromagnetism. In fact, all magnetic fields are solenoidal).

Proposition 9.7. *If $\vec{G} = \nabla \times \vec{F}$, then $\nabla \cdot \vec{G} = 0$.*

If $\vec{G}$ be a vector field defined everywhere and $\nabla \cdot \vec{G} = 0$, then $\vec{G}$ is a curl field, that is, there exists a vector field $\vec{F}$ such that $\nabla \times \vec{F} = \vec{G}$.

Proof. Suppose there is a vector field $\vec{F}$ such that $\nabla \times \vec{F} = \vec{G}$. Then at any point, we know that

$$\nabla \cdot \vec{G} = \lim \frac{\int_S \vec{G} \cdot d\vec{A}}{\text{Volume}}.$$

But by Stokes's theorem, we know that

$$\int_S \vec{G} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}$$

where C is the boundary of S. But S is a sphere, so the boundary is empty, and this integral must be zero.

(You can also draw an arbitrary closed curve along S, and compute the flux integrals of the two pieces S is divided into; they must be equal except for sign, again by Stokes's theorem).

Then

$$\nabla \cdot \vec{G} = \lim \frac{0}{\text{Volume}} = \lim 0 = 0.$$

Conversely, suppose that $\nabla \cdot \vec{G} = 0$. Then we define a function

$$\vec{F}(\vec{r}) = \int_0^1 \vec{G}(t\vec{r}) \times t\vec{r}\, dt.$$

This function is defined everywhere, and after some annoying algebra (and the knowledge that the curl operator commutes with integrals) we can check that $\nabla \times \vec{F} = \vec{G}$. Notice that here we need $\vec{G}$ to be defined everywhere, in order for this integral to be consistently defined. □

Remark 9.8. Compare both this proposition and its proof with proposition 7.36 in section 7.4.

Example 9.9. We saw earlier that $\vec{G}(x, y) = -y\vec{i} + x\vec{j}$ is solenoidal. Let's find its vector potential.

We know that $\nabla \times \vec{F} = \vec{G}$. In particular, this means that

$$\begin{aligned} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= -y \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= 0. \end{aligned}$$

It's hard to figure out where to start with this, because we have a lot of information to work with.

In fact we have a lot of degrees of freedom, since if f is any scalar function then $\nabla \times (\nabla f) = \vec{0}$ and thus $\nabla \times (\vec{F} + \nabla f) = \vec{G}$, so there are infinitely many options. By choosing a f so that $\frac{\partial f}{\partial z} = -F_3$ we can assume that $F_3 = 0$, and our equations reduce to

$$\begin{aligned} -\frac{\partial F_2}{\partial z} &= -y \\ \frac{\partial F_1}{\partial z} &= x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= 0. \end{aligned}$$

Then we have $F_2(x, y, z) = yz + g(x, y)$, and $F_1(x, y, z) = xz + h(x, y)$. Then the third equation tells us that $\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}$, and so we can take $\vec{F}(x, y, z) = xz\vec{i} + yz\vec{j}$.

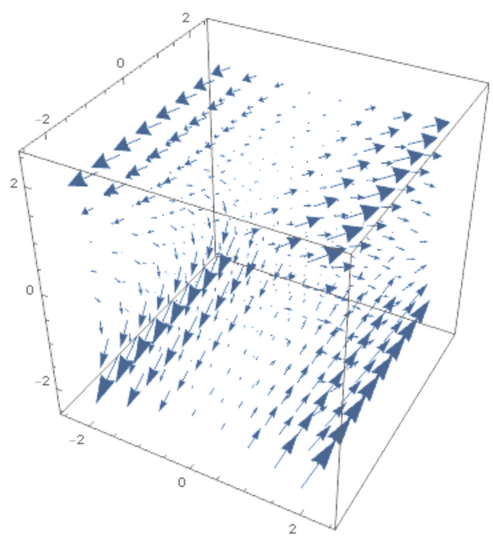
We can check that

$$\nabla \times (xz\vec{i} + yz\vec{j}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & 0 \end{vmatrix} = -y\vec{i} + x\vec{j} + (0-0)\vec{k} = \vec{G}(x,y,z).$$

Example 9.10. Let $\vec{G}(x,y,z) = (x^2, 3xz^2, -2xz)$. Then $\nabla \cdot \vec{G} = 2x + 0 - 2x = 0$, so $\vec{G}$ is solenoidal. We can work out that $\vec{F} = \nabla \times (xz^3, -x^2 z, 0)$ by computing

$$\begin{aligned} -\frac{\partial F_2}{\partial z} &= x^2 \\ \frac{\partial F_1}{\partial z} &= 3xz^2 \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= -2xz. \end{aligned}$$

Then $F_2(x,y,z) = -x^2 z + f(x,y)$ and $F_1(x,y,z) = xz^3 + g(x,y)$. The third equation gives us that $-2xz + \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} = -2xz$ so we can take $f = g = 0$.



9.2 The Divergence Theorem

Now we're ready to state the higher-dimension analogue of Green's Theorem.

Theorem 9.11 (Divergence Theorem). *Let W be a solid three-dimensional region whose boundary S is a piecewise smooth surface oriented outwards, and $\vec{F}$ a smooth vector field on an open region containing S and W. Then*

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \nabla \cdot \vec{F}\, dV.$$

Proof. This proof is basically the same as the proof of Green's theorem.

The left-hand integral is the flux out of the entire boundary. We can approximate the right-hand integral by dividing the region up into small cubes; the divergence in each cube is approximately the average flux out of that cube. Taking the integral adds up the flux from each cube, and we get the total flux out of W. $\square$

Example 9.12. Let $W = \{(x, y, z) : -1 \leq x, y, z \leq 1\}$ be a cube with side length 2 centered at the origin, and let S be its boundary. Set $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$. What is the flux of $\vec{F}$ out of S?

Computing the flux integral directly would involve parametrizing six separate surfaces. Instead we can compute $\nabla \cdot \vec{F} = 1 + 1 + 1 = 3$, so

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \nabla \cdot \vec{F}\, dV = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 3\, dx\, dy\, dz = 24.$$

Suppose instead the vector field is $\vec{F}(x, y, z) = xy\vec{i} + yz\vec{j} + xyz\vec{k}$. Then we compute $\nabla\vec{F}(x, y, z) = y + z + xy$, and our total flux is

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_W \nabla \cdot \vec{F}\, dV \\ &= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 y + z + xy\, dx\, dy\, dz = 0.\end{aligned}$$

Example 9.13. Compute $\int_S \vec{F} \cdot d\vec{A}$ where $\vec{F}(x, y, z) = (x^2yz + y^2z)\vec{i} + xy^2z\vec{j} + (x^2 + y^2)\vec{k}$, and S is the surface of the portion of the unit sphere in the octant $x, y, z \geq 0$. Notice this surface has four pieces!

We could parametrize each piece and compute the surface integral over it, but that seems difficult. Instead we compute

$$\begin{aligned}\nabla \cdot \vec{F}(x, y, z) &= 2xyz + 2xyz + 0 = 4xyz \\ \int_S \vec{F} \cdot d\vec{A} &= \int_W 4xyz \cdot dV \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} 4(\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi)\rho^2\sin\phi\, d\phi\, d\theta\, d\rho \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} 4\rho^4\cos\theta\sin\theta\sin^3\phi\cos\phi\, d\phi\, d\theta\, d\rho \\ &= \int_0^1 \int_0^{2\pi} \rho^4\cos\theta\sin\theta\sin^4\phi|_0^{\pi}\, d\theta\, d\rho = 0.\end{aligned}$$

Proposition 9.14. *If $\vec{F}$ is a solenoidal vector field defined on W, and S is the boundary of W, then $\int_S \vec{F} \cdot d\vec{A} = 0$.*

Proof.

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \nabla \cdot \vec{F}\, dV = \int_W 0\, dV = 0.$$

□

Example 9.15. Let $\vec{F}(\vec{r}) = \frac{\vec{r}}{\|\vec{r}\|^3}$. Let's compute the surface integral of $\vec{F}$ over S, the ellipsoid $x^2 + 4y^2 + 9z^2 = 25$ oriented outwards.

We don't want to parametrize this. But it's easy to compute the flux over the sphere oriented outwards. We see that $\vec{F}$ is always perpendicular to the sphere, so the surface integral is $\|\vec{F}\| \cdot 4\pi = 4\pi$.

Now consider the region of space W bounded on the inside by a sphere of radius 1, and bounded on the outside by S. Then by the divergence theorem, we have

$$\int_W \nabla \cdot F\, dV = \int_{S-T} \vec{F} \cdot d\vec{A}$$

where T is the unit sphere oriented outwards; we have the integral over $S - T$ because we want the boundary oriented away from the region, and thus we take the sphere oriented inwards.

But we saw in example 9.6 that $\nabla \cdot \vec{F}(\vec{R}) = 0$. Thus we have

$$\begin{aligned} 0 &= \int_{S-T} \vec{F} \cdot d\vec{A} \\ &= \int_S \vec{F} \cdot d\vec{A} - \int_T \vec{F} \cdot d\vec{A} \\ \int_S \vec{F} \cdot d\vec{A} &= \int_T \vec{F} \cdot d\vec{A} = 4\pi. \end{aligned}$$

Thus the flux integral through S is equal to 4π.

9.3 The Three Theorems

Over this course, we have defined three different derivative operators:

- The gradient ∇ takes in a scalar function and outputs a vector field.
- The curl $\nabla\times$ takes in a vector field and outputs another vector field.
- The divergence $\nabla\cdot$ takes in a vector field and outpus a scalar function.

We also discovered the following relationships among the derivative operators:

1. $\nabla \times \nabla f = \vec{0}$

2. If $\vec{F}$ is defined everywhere and $\nabla \times \vec{F} = \vec{0}$, then there is a f with $\nabla f = \vec{F}$.

3. $\nabla \cdot \nabla \times G = \vec{0}$

4. If $\vec{G}$ is defined everywhere and $\nabla \cdot \vec{G} = 0$ then there is a $\vec{F}$ with $\nabla \times \vec{F} = \vec{G}$.

Notice that the first two statements look just like the last two statements. Whenever we have a collection of theorems like this that all look similar, we should ask what they're a specialcase of.

We also developed three major theorems that let us convert one integral into another.

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$$
$$\int_S \nabla \times \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r}$$
$$\int_W \nabla \cdot \vec{F}\, dV = \int_S \vec{F} \cdot d\vec{A}$$

Again, these theorems all look similar: they each tell us that the integral over a region of the derivative of a function, is equal to the integral over the boundary of the original function. Again, we want to figure out the right statement to generalize all these theorems. The correct statement requires us to understand differential forms.

9.4 Differential Forms

differential equation (indefinite integral); Lebesgue integral (unsigned definite integral); integration of forms (signed definite integral).

How does an integral work? Let's think about the single-variable case. If we're doing a path integral over a path r, we compute something like

$$\int_r \vec{F} \cdot d\vec{r} = \lim \sum \vec{F}(\vec{x}_i) \cdot \vec{r}\,'(\vec{x}_i)\Delta x.$$

What we're doing here is computing the infinitesimal work over a very small straight-line movement, and then adding up all of these infinitesimals. In order to do this, we need two things: we need a path to integrate over, and we need some way of computing the work done over a small section of that path. We want to generalize both of those things.

Definition 9.16. A k-dimensional *parametrized oriented manifold* in an ambient space $\mathbb{R}^n$ is a function $\vec{r} : [0,1]^k \to \mathbb{R}^n$. An *oriented manifold* is any surface that can be broken into pieces, each of which can be parametrized in this way.

Remark 9.17. In practice we don't always choose parametrizations where all of our bounds go from 0 to 1. But we could if we wanted to, and the theory is easier to work with if we make that assumption. It doesn't change anything important.

In the one-dimensional case, we divided our curve up into infinitesimal line segments with lengths Δx. In the k-dimensional case we want to chop things up into infinitesimal squares or cubes or hypercubes. We will represent an infinitesimal square with $\Delta x_1 \wedge \Delta x_2$, a cube with $\Delta x_1 \wedge \Delta x_2 \wedge \Delta x_3$, and so on. (The wedge represents an "exterior product", but don't worry too much about what that means).

In our path integral, we needed some function that would take in an infinitesimal path, at a location in space, and tell us how much work was done by moving over that infinitesimal path. Thus, we needed a function that takes in a point $\vec{x}$ and an infinitesimal vector $\Delta\vec{x}$, and outputs an amount of work. Thus we need a function $\omega_{\vec{x}} : \mathbb{R}^n \to \mathbb{R}$.

Further, we want this function to be *linear*, which means that it commutes with addition and multiplication:

1. $\omega_{\vec{x}}(\Delta\vec{x}_1 + \Delta\vec{x}_2) = \omega_{\vec{x}}(\Delta\vec{x}_1) + \omega_p(\Delta\vec{x}_2)$
2. $\omega_{\vec{x}}(r\Delta\vec{x}) = r\omega_{\vec{x}}(\Delta\vec{x})$.

So where can we get these functions from? It turns out that every function with this property can be given by a dot product, and there is some $\vec{F}$ such that $\omega_{\vec{x}}(\vec{v}) = \vec{F}(\vec{x}) \cdot \vec{v}$. Thus we have the usual path integral formulation:

$$\int_r \omega \approx \sum \omega_{\vec{x}_i}(\Delta\vec{x}_i) = \sum \vec{F}(\vec{x}_i) \cdot \Delta\vec{x}_i \approx \sum \vec{F}(\vec{x}_i) \cdot \vec{r}\,'(\vec{x}_i)\Delta x_i \approx \int_r F \cdot d\vec{r}.$$

We'd like to generalize this idea, to the idea of a *differential form.*

Definition 9.18. A k-form at a point $\vec{x}$ is a multilinear function $\omega_{\vec{x}} : (\mathbb{R}^n)^k$ to $\mathbb{R}$. That is, a k form takes in k vectors and outputs a real number, and satisfies the axioms

1. $\omega_{\vec{x}}(\Delta\vec{x}_1 \wedge \cdots \wedge \Delta\vec{x}_k) + \omega_{\vec{x}}(\Delta\vec{y}_1 \wedge \cdots \wedge \Delta\vec{x}_k) = \omega_{\vec{x}}((\Delta\vec{x}_1 + \Delta\vec{y}_1) \wedge \cdots \wedge \Delta\vec{x}_k)$
2. $r\omega_{\vec{x}}(\Delta\vec{x}_1 \wedge \cdots \wedge \Delta\vec{x}_k) = \omega_{\vec{x}}(r\Delta\vec{x}_1 \wedge \cdots \wedge r\Delta\vec{x}_k)$

We also require that $\omega_{\vec{x}}\Delta\vec{x} \wedge \Delta\vec{x} = 0$ for any $\vec{x}$. This represents the idea that a "parallelogram" whose two edges are given by the same vector has zero area.

Remark 9.19. We can also define a zero-form, which at each point takes in no vectors at all and outputs a real number. A zero-form is then just a scalar function.

We can get one more fact out of this wedge product: we have

$$\begin{aligned}
0 &= (\Delta\vec{x}_1 + \Delta\vec{x}_2) \wedge (\Delta\vec{x}_1 + \Delta\vec{x}_2) \\
&= \Delta\vec{x}_1 \wedge \Delta\vec{x}_1 + \Delta\vec{x}_1 \wedge \Delta\vec{x}_2 + \Delta\vec{x}_2 \wedge \Delta\vec{x}_1 + \Delta\vec{x}_2 \wedge \Delta\vec{x}_2 \\
&= \Delta\vec{x}_1 \wedge \Delta\vec{x}_2 + \Delta\vec{x}_2 \wedge \Delta\vec{x}_1 \\
\Delta\vec{x}_1 \wedge \Delta\vec{x}_2 &= -\Delta\vec{x}_2 \wedge \Delta\vec{x}_1.
\end{aligned}$$

Thus the wedge product is anticommutative.

Every differential k-form can be written $f(\vec{x})dx_{a_1} \wedge \cdots \wedge dx_{a_k}$ for some scalar function $f(\vec{x})$ and some $a_i \in \{1, \ldots, n\}$. (In algebraic language, the set $\{dx_{a_1} \wedge \cdots \wedge dx_{a_k}\}$ forms a basis for the space of differential k-forms).

In $\mathbb{R}^3$, a 1-form is $f dx + g dy + h dz$; we have actually seen this notation in section 8.1 when we talked about differential notation. Similarly, a 2-form is $f dxdy + g dxdz + h dydz$. Every 3-form is $f dxdydx$.

Definition 9.20. We define the *integral* of a k-form ω over a k-dimensional oriented manifold r by

$$\int_r \omega = \sum \omega_{\vec{x}_i}(\Delta\vec{x}_{i,1} \wedge \ldots \Delta\vec{x}_{i,k}).$$

We've introduced a bunch of new notation now. What does that get us? Well, first, we've figured out how to extend our definitions to integrals in more than three dimensions, at least sort of. But it actually gets us a lot more after we define the derivative.

If f is a scalar function, then the derivative df at the point $\vec{x}$ is a linear function such that $f(\vec{x} + \vec{v}) \approx f(\vec{x}) + df_{\vec{x}}(\vec{v})$. (Algebraically this is given by the gradient). But this is a linear function that takes in a vector and outputs a scalar, and thus is a 1-form. We can generalize this:

Definition 9.21. Let $\omega = f dx_{a_1} \wedge \cdots \wedge dx_{a_k}$ be a k-form. Then we define the derivative of ω to be the $k+1$-form

$$d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{a_1} \wedge \cdots \wedge dx_{a_k}.$$

Proposition 9.22. *Let ω be a k-form and α a ℓ-form. Then*

1. $d(d\omega) = 0$
2. $d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^k(\omega \wedge d\alpha)$.

We now can explain all of the patterns we saw in section 9.3. We have already seen that 0-forms are scalars, and we can identify 1-forms with vector fields. When the ambient space is $\mathbb{R}^3$, we can also identify 2-forms with vector fields, and 3-forms with scalar functions. (For instance, a 3-form in $\mathbb{R}^3$ is always given by $f dx \wedge dy \wedge dz$, so we can identify it purely by the function f).

We had three derivatives: the gradient, the curl, and the divergence. In $\mathbb{R}^3$, then the derivative of a 0-form is given by the gradient; the derivative of a 1-form is given by the curl; and the derivative of a 2-form is given by the divergence.

We had the patterns given by the Fundamental Theorem of Line Integrals, Stokes's theorem, and the Divergence Theorem. These are all generalized into the generalized Stokes's theorem.

Theorem 9.23 (Stokes)**.** *If M is an oriented $k+1$-dimensional manifold with boundary ∂M an oriented k-dimensional manifold, and ω is a k-form, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This makes it important to ask when a $(k+1)$-form is the derivative of some k-form.

Definition 9.24. We say a k-form ω is *closed* if $d\omega = 0$. A closed 1-form is irrotational and a closed 2-form is solenoidal.

We say that ω is *exact* if there is a $(k-1)$-form α with $d\alpha = \omega$. An exact 1-form is conservative and an exact 2-form is a curl field.

We can see that every exact form must also be closed, since then $d\omega = d(d\alpha) = 0$. It is not the case, that we have seen in class, that every closed form is exact; however, a closed form defined everywhere on Euclidean space must be exact.

More generally, every closed form on a domain with no holes is exact; but a domain with holes in it will have closed forms that are not exact. This means we can use the difference between closed forms and exact forms to measure the extent to which our domain has holes.

Definition 9.25. If α, β are closed k-forms on a manifold M, then we say that α and β are *cohomologous* if $\alpha - \beta$ is an exact form. This is an equivalence relation on closed k-forms.

We define the kth *de Rham cohomology group* of M, written $H^k_{dR}(M)$, to be the set of equivalence classes of closed k-forms under the cohomology relation. In fact this set forms a *group* under the operation of addition.

If $H^k_{dR}(M)$ contains exactly one element, then all closed forms are exact. This implies that every k-dimensional sphere can be contracted through M to become a single point.

$H^1_{dR}(M)$ contains exactly one element, we say that M is *simply connected*.

This doesn't seem that useful, but the theory of algebraic topology and the Mayer-Vietoris sequence allows us to compute these cohomology groups in a relatively easy and calculus-free way. Unfortunately, We won't be discussing that here.

Finally, we should mention the pull-back. Suppose M and N are manifolds and $\phi : M \to N$ is a function between them. If we have a k-form ω defined on N, then we can use it to define a form $\phi^*(\omega)$ on M that satisfies

$$\int_{\phi(M)} \omega = \int_M \phi^*(\omega).$$

We call this form the *pull-back* of ω along ϕ, and define it essentially by plugging points of M into ϕ before plugging that into ω; thus if f is a 0-form, we have $\phi^* f(\vec{x}) = f(\phi(\vec{x}))$; and if ω is a 1-form we have

$$(\phi^*\omega)_{\vec{x}}(\vec{v}) = \omega_{\phi(\vec{x})}(\phi(\vec{v})).$$

The pull-back conveniently satisfies the relationships $\phi^*(\omega \wedge \alpha) = (\phi^*\omega) \wedge (\phi^*\alpha)$ and $d(\phi^*\omega) = \phi^*(d\omega)$; and from the pull-back, we can recover the change-of-variable integral formulas we use for u-substitution and in section 6.3.