Math 212 Final Exam

Instructor: Jay Daigle

- Please try to take less than four hours for this test. The test is intended to take about three hours.

  If you want to keep going after four hours, please change pen colors or something and make it clear where the four hours end.
- This test is open notes. You may use your notes, and anything I have posted to the website. Please don't use any other sources or materials.
- You may use a normal or scientific calculator. You may not use a graphing calculator. A calculator is not required to complete this exam. Please don't use any interesting computational software or anything that can compute derivatives or integrals for you.
- Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all.
- I will do my best to monitor Zoom during the afternoons and evenings to answer questions, but I can't make any promises. I will actively monitor Zoom during the official final exam time of Friday, May 8, 1-4 PM.
- Good luck!

Name:

Time Started:

Time Completed:

1		5	
2		6	
3		7	
4		8	
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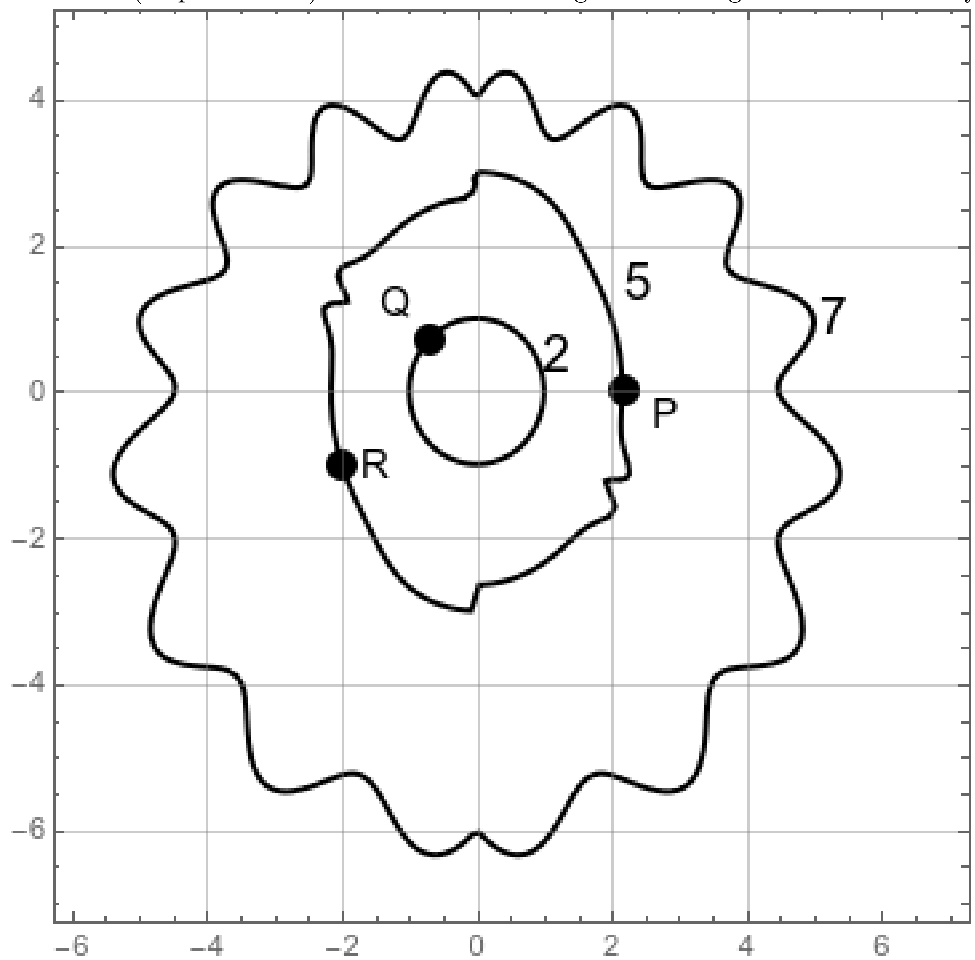
Problem 1. (10 points each) Consider the following contour diagram for a function $f(x, y)$.



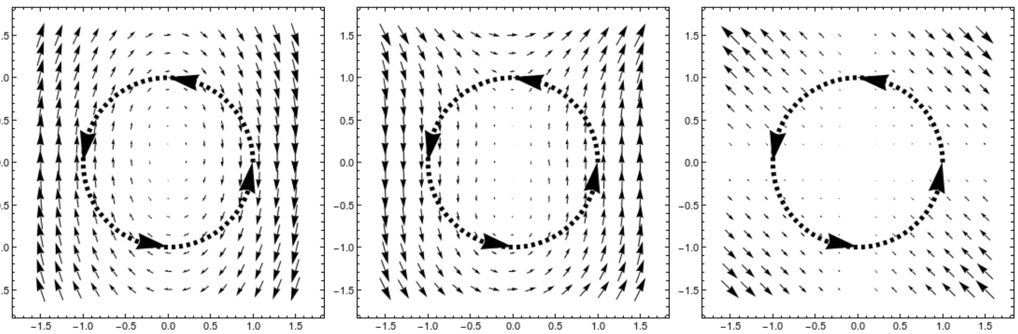
(a) Estimate $\frac{\partial f}{\partial x}(P)$ and briefly explain your reasoning.

(b) Estimate the directional derivative in the direction $\vec{j} - \vec{i}$ at Q, and briefly explain your reasoning.

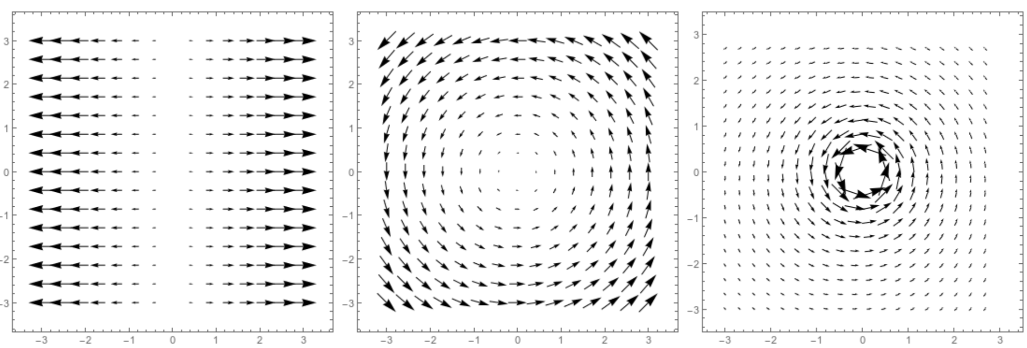
(c) Draw a vector representing the gradient at point R. Briefly explain how you know this is the gradient.

Problem 2. (10 points each)

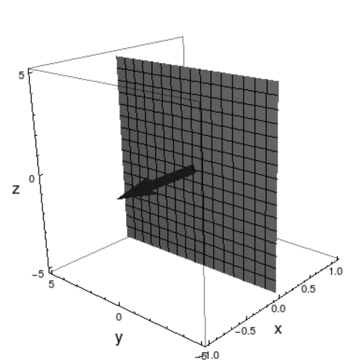
(a) For each of the following pictures, is the integral of the given vector field over the given curve positive, negative, or zero? Briefly explain your reasoning for each.

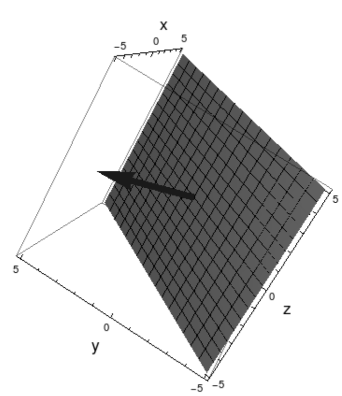


(b) Which of the following vector fields is conservative? (There is exactly one). Briefly explain your reasoning.



(c) Let $\vec{F}(x, y, z) = \vec{i} + \vec{j}$. For each of the following oriented surfaces, is the flux of $\vec{F}$ through it positive, negative, or zero? Briefly explain your reasoning for each.





Problem 3. (15 points each)

(a) Find a linear approximation of $f(x, y) = \sin(x)\sqrt{1 - y^2}$ near the point $(0, 0)$. Use it to estimate $f(.1, .1)$.

(b) Find and classify all the critical points of $g(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$.

(c) Find the minimum value of $f(x, y) = 4xy$ on the unit circle.

Problem 4. (15 points each) Let

$$\vec{F}(x, y, z) = (0, x, y) \qquad \vec{G}(x, y, z) = (2x, z, y) \qquad \vec{H}(x, y, z) = (3y, 2x, z).$$

(a) For each field, either find a scalar potential function or prove that none exists.

(b) For each field, either find a vector potential function or prove that none exists.

(c) Let $\vec{r}(t) = (2, 2t, t^2)$. For which of these vector fields is $\vec{r}$ a flow line? Justify your answer.

Problem 5. (10 points each) Let $f(x, y) = x^2 + y^2 + x$ and let R be the right half of the unit circle.

(a) Set up integrals to compute $\int_R f\,dA$ in both cartesian and polar coordinates.

(b) Choose one of the integrals from part (a) and evaluate it.

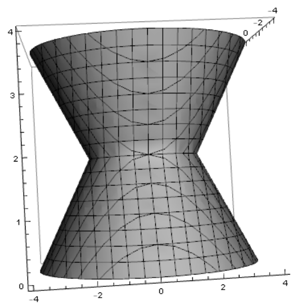
Problem 6. (15 points each) Let $g(x, y, z) = z(x^2 + y^2)$ and let W be a cone with its point at the origin and base given by the circle $z = 2, x^2 + y^2 = 2$.

(a) Set up integrals to compute $\int_W g\,dV$ in cartesian, cylindrical, and spherical coordinates.

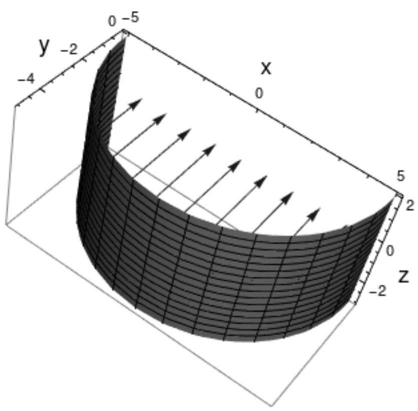
(b) Choose one of the integrals from part (a) and evaluate it.

Problem 7. (10 points each) Set up but **do not compute** an integral to answer each of the following questions. Each answer should be an iterated integral containing no vector operations and no variables other than the variables of integration.

(a) Find the volume of the following shape made up of two cones squashed together, which has its base at $z = 0$, its top at $z = 4$, and has a radius of 4 at the base and top, and a radius of 2 at the thinnest point at $z = 2$.



(b) What is the flux of the vector field $\vec{F}(x, y, z) = xy\vec{i} + xz\vec{j} + yz\vec{k}$ through the $y \leq 0$ half of the side of a cylinder of radius 5, centered at the z axis, which goes from $z = -3$ to $z = 2$, oriented towards the z-axis?

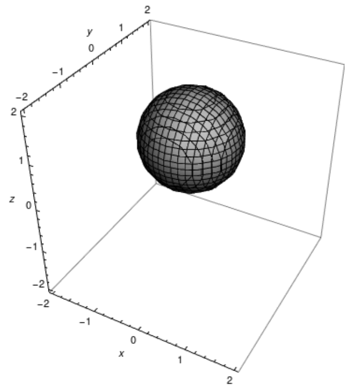


(c) What is the work done by the force field $\vec{G}(x,y,z) = \sin(xz)y\vec{i} + e^{xyz}\vec{j} + \sqrt{x+y+z}\vec{k}$ on a particle following the path $\vec{r}(t) = (t, t^2, t^4)$ from time $t = 0$ to time $t = 5$.

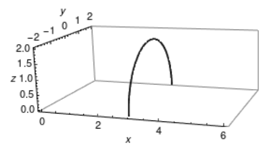
(d) Integrate the function $f(x,y) = 5xy^3$ over the region bounded by $y = 9 - x^2$ and $y = 3 - x$. Sketch the region of integration.

(e) What is the surface area of the graph of $f(x,y) = e^{xy} + \sin(x)\cos(y)$ for $0 \le x \le 3$ and $1 \le y \le \pi$?

(f) Find the mass of a solid spherical ball of radius 1 centered at the point $(0, 0, 1)$ if its density is given by $\delta(x, y, z) = x^2 z$.

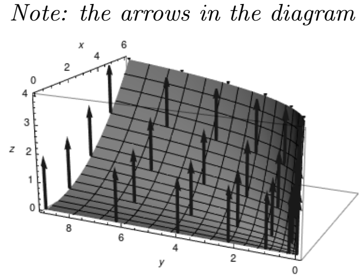


(g) What is the mass of a wire following a semi-circular path of radius 2 contained in the $x = 3$ plane, which goes from $(3, 2, 0)$ through $(3, 0, -2)$ to $(3, -2, 0)$, with density given by $\delta(x, y, z) = x^2 + y^2 + \sqrt{z^2 + 1}$?



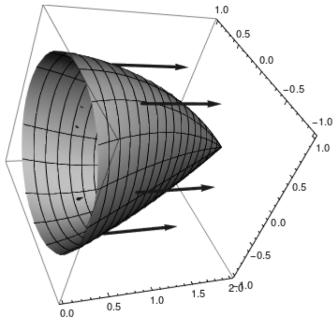
(h) Find the flux of the vector field $\vec{F}(x, y, z) = (x, xy, z)$ through the surface parametrized by $\vec{r}(s, t) = (st, s^2, t^2)$ oriented upwards, for $0 \leq s \leq 3, 0 \leq t \leq 2$.

Note: the arrows in the diagram are the orientation of the surface, not a representation of F.

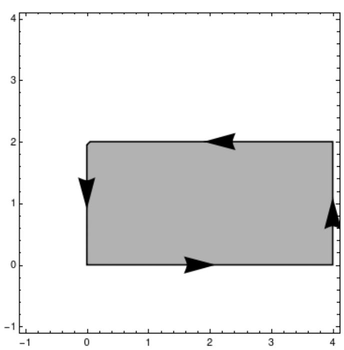


Problem 8. (20 points each) Compute each of the following integrals. You may often wish to use a theorem or other result to replace the given integral with an easier integral. Please identify the result you are using.

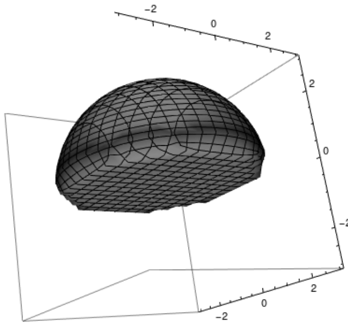
(a) Let $\vec{F}(x, y, z) = \sqrt{x^5 + x}\vec{i} + (x^2yz - z)\vec{j} + (x\sqrt{z^3 + y} + y)\vec{k}$. Compute the flux of the vector field $\nabla \times F$ through a net whose rim is the unit circle $y^2 + z^2 = 1$ in the $x = 0$ plane, oriented in the $\vec{i}$ direction.



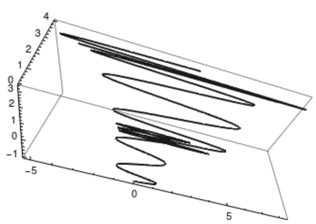
(b) Find the circulation of $\vec{F}(x, y) = -3y\vec{i} + 2x\vec{j}$ counterclockwise around the rectangle $0 \le x \le 4, 0 \le y \le 2$.



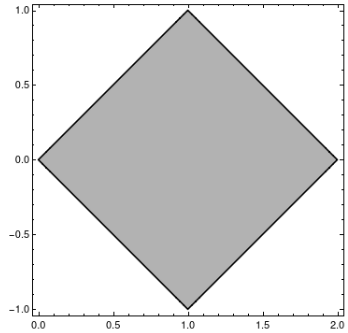
(c) Integrate the function $f(x, y, z) = z$ over the $z \geq 0$ half of the solid radius-3 spherical ball centered at the origin.



(d) Find the integral of the vector field $\vec{F}(x, y, z) = yz\vec{i} + xz\vec{j} + xy\vec{k}$ over the path $\vec{r}(t) = (t + \sin(10\pi t)e^t, t^2 - \cos(2\pi t), 2^t)$ as t varies from 0 to 2.



(e) Integrate the function $(x+y)(x-y)$ over the diamond bounded by $x+y=2, x+y=0, x-y=0, x-y=2$.



(f) Compute $\int_S \vec{F} \cdot d\vec{A}$, where $\vec{F}(x,y,z) = xy^2\vec{i} + x^2y\vec{j} + x^2y^2\vec{k}$ and S is the surface (including both ends!) of a closed cylinder with radius 2 centered on the z-axis, from $z=-2$ to $z=2$.