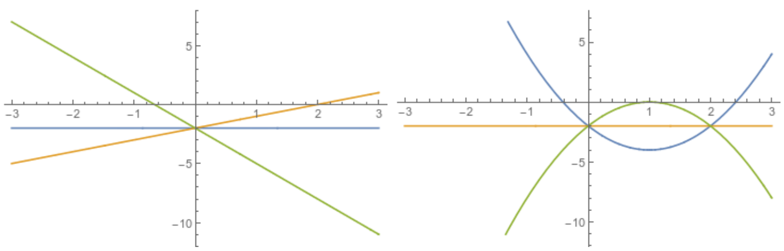
Math 212 Test 1 Solutions

Instructor: Jay Daigle

Problem 1. Let $f(x,y) = 2xy - x^2y - 2$

(a) Sketch cross-sections of f for $x = -1, 0, 1$ and $y = -2, 0, 2$.

Solution:



$$f(-1,y) = -2y - y - 2 = -3y - 2 \qquad f(0,y) = -2 \qquad f(1,y) = 2y - y - 2 = y - 2$$
$$f(x,-2) = -4x + 2x^2 - 2 \qquad f(x,0) = -2 \qquad f(x,2) = 4x - 2x^2 - 2$$

(b) If $\vec{u} = \frac{-3}{5}\vec{i} + \frac{4}{5}\vec{j}$, compute $f_{\vec{u}}(2,1)$.

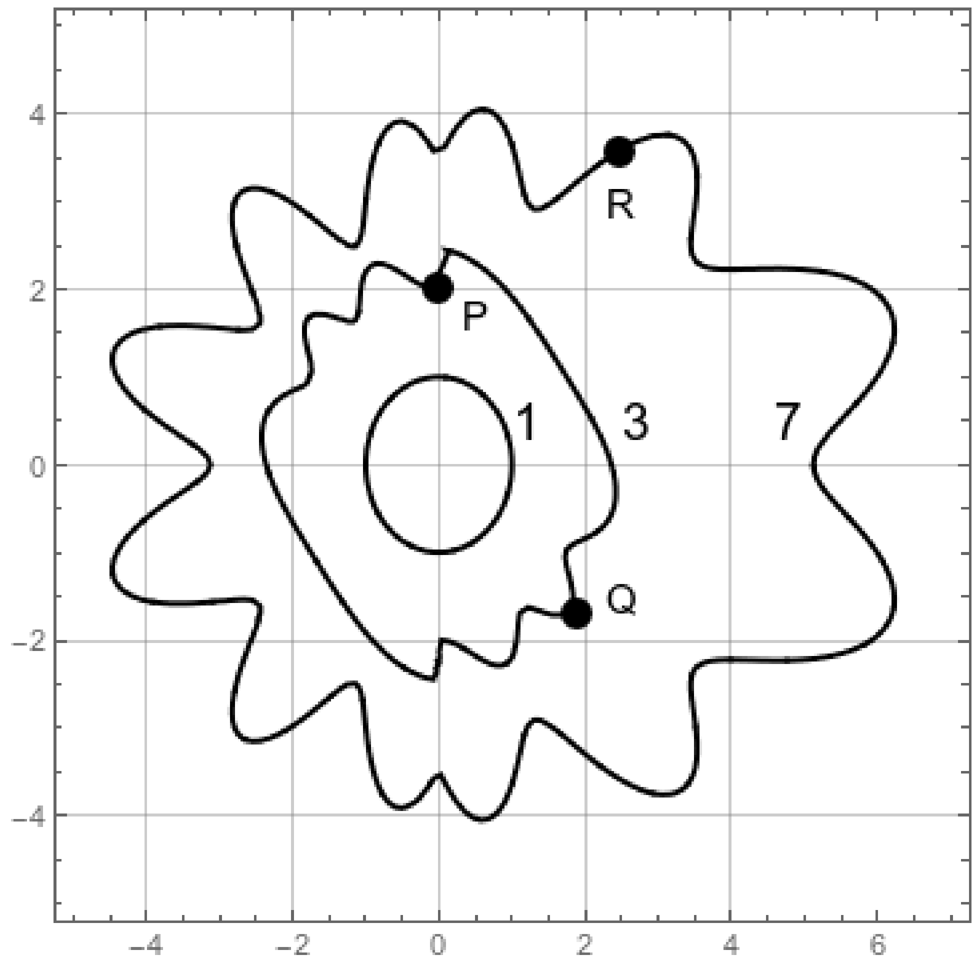
Solution: $\nabla f(x,y) = (2y - 2xy)\vec{i} + (2x - x^2)\vec{j}$, so

$$\nabla f(2,1) = -2\vec{i}$$
$$f_{\vec{u}}(1,1) = 6/5.$$

(c) At the point $(3,1)$, in what direction should you move to increase $f(x,y)$ at the fastest possible rate? What is the rate of increase in that direction?

Solution: The direction of greatest increase is $\nabla f(3,2) = -4\vec{i} - 3\vec{j}$. The rate of increase is $\|\nabla f(3,2)\| = \sqrt{(-4)^2 + (-3)^2} = 5$.

Problem 2. Here is a contour graph for a function $g(x,y)$.



(a) Estimate $\frac{\partial g}{\partial y}$ at the point P and explain your reasoning.

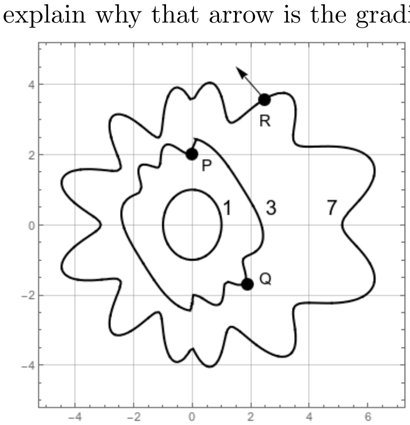
Solution: It should be a little bit greater than 2. We see that going from $y = 1$ to $y = 2$ the value increases by 2, and going from $y = 2$ to $y = 4$ the value increases a little bit more than 4. So the estimate is a bit over 2, maybe 2.2 or 2.5.

(b) Is the sign of the directional derivative of g at the point Q in the direction $\vec{i} - \vec{j}$ positive, negative, or zero? Explain your reasoning.

Solution: Positive, since we're going from the $g = 3$ contour towards the $g = 7$ contour.

(c) On the contour diagram, sketch and clearly label an arrow in the direction of the gradient at R. In a few words explain why that arrow is the gradient.

Solution:



Problem 3. (a) Give an equation for a plane through the points $(1,1,1),(1,3,5),(3,1,-3)$.

Solution: There are two approaches here.

First, we can observe that the first two points share a x coordinate and the first and third share an y coordinate. Thus we can compute the x slope is -2 and the y slope is 2. Then our equation is

$$z = -2(x-1) + 2(y-1) + 1 = -2x + 2y + 1.$$

Alternatively, we get the vectors $2\vec{j} + 4\vec{k}$ and $2\vec{i} - 4\vec{k}$. Then we compute

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 4 \\ 2 & 0 & -4 \end{vmatrix} = -8\vec{i} + 8\vec{j} - 4\vec{k} = \vec{n}$$

and thus the equation for the plane is

$$0 = -8(x-1) + 8(y-1) - 4(z-1).$$

These are, non-obviously, the same plane.

(b) Find an equation for the plane parallel to the vectors $\vec{i} - 3\vec{j} + \vec{k}$ and $-\vec{i} + 4\vec{j} - 2\vec{k}$ through the point $(1,1,1)$.

Solution: We first compute the cross product:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ -1 & 4 & -2 \end{vmatrix} = 2\vec{i} + \vec{j} + \vec{k}.$$

This will be a normal vector to the plane. Thus the equation for the plane is

$$2(x-1) + 1(y-1) + 1(z-1) = 0.$$

(c) Find the cosine of the angle between the vectors $\vec{v} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{u} = \vec{i} - 2\vec{j} + \vec{k}$.

Solution: We know that

$$\cos\theta = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \cdot \|\vec{u}\|} = \frac{3-4-1}{\sqrt{14}\sqrt{6}} = \frac{-2}{2\sqrt{21}} = \frac{-1}{\sqrt{21}}.$$

Problem 4. (a) Compute $\nabla\left(x^2 z + \sqrt{xy}\right)$.

Solution:

$$(2xz + \frac{1}{2}\sqrt{y/x})\vec{i} + \frac{1}{2}\sqrt{x/y}\vec{j} + x^2\vec{k}$$

(b) Compute $f_{\vec{v}}(-1,2)$ when $f(x,y) = x^3 + \frac{y^2}{x}$ and $\vec{v} = 2\vec{i} - \vec{j}$.

Solution:

$$\begin{aligned} \nabla f &= (3x^2 - \frac{y^2}{x^2})\vec{i} + \frac{2y}{x}\vec{j} \\ \nabla f(-1,2) &= -\vec{i} - 4\vec{j} \\ f_{\vec{v}}(-1,2) &= (-2+4)/\sqrt{5} = \frac{2}{\sqrt{5}}. \end{aligned}$$

(c) Let $\vec{v} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{u} = -2\vec{i} - \vec{j} + 2\vec{k}$. Compute the orthogonal decomposition of $\vec{v}$ with respect to $\vec{u}$. That is, write $\vec{v} = \vec{v}_{\text{paralle}} + \vec{v}_{\perp}$.

Solution:

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{-6 - 1 - 2}{4 + 1 + 4} \vec{u} \\
&= \frac{-9}{9} \vec{u} = 2\vec{i} + \vec{j} - 2\vec{k} \\
\vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = \vec{i} + \vec{k}.
\end{aligned}$$

Problem 5. (a) Find an equation for the tangent plane to the graph of the function $f(x, y) = e^{xy} + x/y$ at the point $(0, 2)$.

Solution: We have $\nabla f(x, y) = (e^{xy}y + 1/y)\vec{i} + (e^{xy}x - x/y^2)\vec{j}$, so $\nabla f(0, 2) = 5/2\vec{i} + 0\vec{j}$.

Further we have $f(0, 2) = 1 + 0$. Thus we get the equation

$$z = 1 + \frac{5}{2}(x - 0).$$

(b) Let $g(x, y, z) = x^2y + y^2z$. Use a linear approximation at the point $(1, 2, 3)$ to estimate $g(.9, 2.1, 3.2)$.
Solution:

$$\begin{aligned}
\nabla g(x, y, z) &= 2xy\vec{i} + (x^2 + 2yz)\vec{j} + y^2\vec{k} \\
\nabla g(1, 2, 3) &= 4\vec{i} + 13\vec{j} + 4\vec{k} \\
g(x, y, z) &\approx 4(x - 1) + 13(y - 2) + 4(z - 3) + 14 \\
g(.9, 2.1, 3.2) &\approx 4(-.1) + 13(.1) + 4(.2) + 14 = -.4 + 1.3 + .8 + 14 = 15.7.
\end{aligned}$$