$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$L : U \to V$

$L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v})$

$L(r\vec{u}) = rL(\vec{u})$

$\mathrm{Im}(L) = \{L(\vec{u})\}$

$\ker(L) = \{\vec{u} \mid L(\vec{u}) = \vec{0}\}$

$L_A(\vec{x}) = A\vec{x}$ is linear

Row vectors $\vec{r}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$

column vectors $c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Dfn: Rowspace

$\mathrm{row}(A) = \mathrm{span}\{\vec{r}_i\}$

Column space

$\mathrm{col}(A) = \mathrm{span}\{\vec{c}_j\}$.

Rank $rk(A) = \dim row(A)$

nullspace $N(A) = \ker(A)$

$= \{\vec{x} \mid A\vec{x} = \vec{0}\}$

$$\begin{array}{ccc} \text{col} & \longleftrightarrow & \text{image} \\ \text{row}^{*} & & \\ \text{null} & \longleftrightarrow & \text{ker} \end{array}$$

Prop: $A \in M_{m \times n}$, $\vec{b} \in \mathbb{R}^m$

$im(A) = col(A)$

$A\vec{x} = \vec{b}$ has a soln iff $b \in col(A)$

Pf/
$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Cor: $A\vec{x}=\vec{b}$ has a soln $\forall \vec{b}\in\mathbb{R}^m$

iff

$col(A)=\mathbb{R}^m$. (cols span $\mathbb{R}^m$)

System has a unique soln

if cols are LI.

Cor: $A \in M_{m\times n}$, A_R the RREF

Then The non-zero rows of A_R form a basis for row(A).

Pf/ non-zero rows of A_R are LI

Span row(A_R) = row(A). //

find basis for row(A)

$$A = \begin{bmatrix} 1 & 5 & -9 & 11 \\ -2 & -9 & 15 & -21 \\ 3 & 17 & 30 & 36 \\ -1 & 2 & -3 & -1 \end{bmatrix}$$

$$\downarrow$$

$$U \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{bmatrix} \right\}$$

Thm (Rank-Nullity): $A \in M_{m \times n}$.

Then $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pf/ A_r is RREF of A

$A\vec{x} = \vec{0}$ iff $A_r\vec{x} = \vec{0}$

rank = # non-zero rows

= # leading 1s.

nullity = # columns w/o leading 1s.

rank + nullity = # columns.

Prop: $A \in M_{m \times n}$.

Then $\dim(\text{row}(A)) = \dim(\text{col}(A))$.

Pf/claim: $\dim(\text{col}(A)) \geq \dim(\text{row}(A))$.

A rank r, U RREF of A.

U has r cols w/ leading 1's, LI

$U_L = U$, delete cols w/o leading 1s.

A_L = delete some cols from A.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A_L, U_L are row-equivalent

$A_L \vec{x} = 0$ iff $U_L \vec{x} = 0$.

iff $\vec{x} = \vec{0}$

So col's(A_L) are LI.

So $\dim(\text{col}(A)) \geq rk(A)$.

$$\dim(\mathrm{col}(A)) \geq \dim(\mathrm{row}(A)).$$

WTS: $=$

$$\begin{array}{ccc} \dim(\mathrm{col}(A^T)) & \geq & \dim(\mathrm{row}(A^T)) \\ \shortparallel & & \shortparallel \\ \dim(\mathrm{row}(A)) & & \dim(\mathrm{col}(A)) \end{array} \quad //$$

$$\begin{aligned} \text{rank} &= \dim(\mathrm{row}(A)) \\ &= \dim(\mathrm{col}(A)) \\ &= \dim(\mathrm{im}(A)). \end{aligned}$$

if wts $S = \{\vec{r}_1, \dots, \vec{r}_n\}$ spans

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vdots \\ r_n \end{bmatrix}$$

rank = dim row(A)

rows span iff rank = dim space

= # columns

Cor: $A \in M_{m \times n}$
U RREF
the columns of A
that correspond
to cols of U w/
leading 1s.
form a basis
for col(A).

Pf/ I know these cols
are LI.
cols is rank(A)
$=\dim \text{col}(A)$.

find basis for col(A)

$$A = \begin{bmatrix} 1 & 5 & -9 & \cancel{11} \\ -2 & -9 & 15 & \cancel{-21} \\ 3 & 17 & 30 & \cancel{36} \\ -1 & 2 & -3 & \cancel{-1} \end{bmatrix}$$

$$\downarrow$$

$$U \begin{bmatrix} 1 & 0 & 0 & \cancel{4} \\ 0 & 1 & 0 & \cancel{2} \\ 0 & 0 & 1 & \cancel{1/3} \\ 0 & 0 & 0 & \cancel{0} \end{bmatrix}$$

$$B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -9 \\ 17 \\ 2 \end{bmatrix}, \begin{bmatrix} -9 \\ 15 \\ 30 \\ -3 \end{bmatrix} \right\}$$

Find bases for row, col, nullspaces of

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

rank $= 3$, nullity $= 2$

$$\text{rowspace} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \Bigg| \quad \text{colspace} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 5 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

rank $= 3$ nullity $= 2$

$$\text{nullspace} = \left\{ \begin{bmatrix} -3x_3 - 7x_4 \\ -x_3 - 3x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} \right\}$$

$$B = \left\{ \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{array}{c} x_3 = 1 \\ x_4 = 0 \end{array} & \begin{array}{c} x_3 = 0 \\ x_4 = 1 \end{array} \end{array} \right\}$$