$L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$

$L(r\vec{x}) = r\,L(\vec{x})$

ker(L), im(L)

---

matrices

colspace $\longleftrightarrow$ image

s

rowspace

nullspace $\longleftrightarrow$ kernel

colspace + nullspace = dim domain

$$A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+5y+z \\ 2x-y+3z \end{bmatrix}$$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$L(\vec{e}_1) = A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad L(\vec{e}_2) = A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$L(\vec{e}_3) = A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Prop: $L: \mathbb{R}^n \to \mathbb{R}^m$. $\exists A \in M_{m\times n}$ s.t. $A\vec{x} = L(\vec{x})\ \forall \vec{x} \in \mathbb{R}^n$

inparticular $\vec{c}_j = L(\vec{e}_j)$

Pf/ Let $A = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n] = [L(\vec{e}_1)\ L(\vec{e}_2) \cdots L(\vec{e}_n)]$

claim $L(\vec{x}) = A\vec{x}\ \forall x \in \mathbb{R}^n$

first: show $L(\vec{e}_j) = A\vec{e}_j$

$$A\vec{e}_j = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n]\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{c}_1 \cdot 0 + \vec{c}_2 \cdot 0 + \cdots + \vec{c}_j \cdot 1 + \cdots + \vec{c}_n \cdot 0$$

$$= \vec{c}_j = L(\vec{e}_j)$$

Now let $\vec{x} \in \mathbb{R}^n$ WTS $A\vec{x} = L(\vec{x})$

Can write $\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i$ $\qquad \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$A\vec{x} = A\left(\sum_{i=1}^{n} x_i \vec{e}_i\right) = \sum_{i=1}^{n} A x_i \vec{e}_i = \sum_{i=1}^{n} x_i A\vec{e}_i = \sum_{i=1}^{n} x_i L(\vec{e}_i)$$

$$\|$$

$$L(\vec{x}) = L\left(\sum_{i=1}^{n} x_i \vec{e}_i\right) = \sum_{i=1}^{n} L(x_i \vec{e}_i) = \sum_{i=1}^{n} x_i L(\vec{e}_i)$$

QED

Cor: if $L, T: \mathbb{R}^m \to \mathbb{R}^n$ and $L(\vec{e}_i) = T(\vec{e}_i)$,
then $L(\vec{x}) = T(\vec{x})$ $\forall x \in \mathbb{R}^m$.

Example: $90°$ rotation $R_{90}: \mathbb{R}^2 \to \mathbb{R}^2$

$$R_{90}(\vec{e}_1) = R_{90}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$R_{90}(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

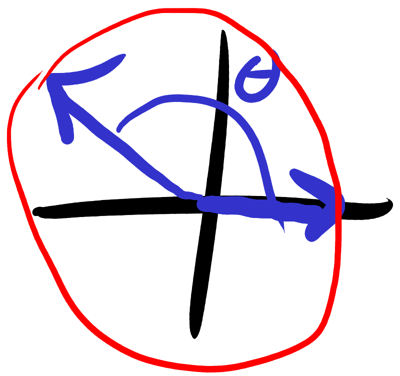
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

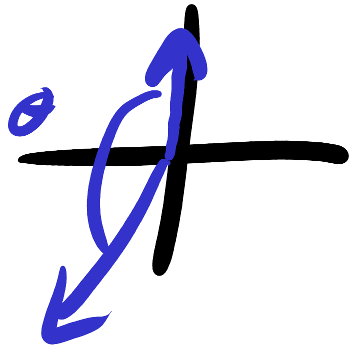
$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotates θ counterclockwise

$$R_\theta(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$R_\theta(\vec{e}_2) = \begin{bmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$





$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$1+3x+4x^2-x^3 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \end{bmatrix}$??

Dfn: U VS

$E = \{\vec{e}_1, \dots, \vec{e}_n\}$ basis

$\vec{u} \in U$.

we can write

$\vec{u} = a_1\vec{e}_1 + a_2\vec{e}_2 + \dots + a_n\vec{e}_n$

unique

Define

$$[u]_E = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

is the coordinate vector of $\vec{u}$.

$E = \{1, x, x^2, x^3\}$

$$[1+3x+4x^2-x^3]_E = \begin{bmatrix} 1 \\ 3 \\ 4 \\ -1 \end{bmatrix}.$$

$U = P_3$, $E = \{1, x, x^2, x^3\}$ $F = \{1, 1+x, 1+x^2, 1+x^3\}$

$f(x) = 1 + 3x + x^2 - x^3 \in U$

$[f]_E = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$ $[f]_F = \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \end{bmatrix}$

$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$

$f(x) = (-1)(1+x^3) + 1(1+x^2) + 3(1+x)$
$-2(1)$

$U = \mathbb{R}^3, \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad [\vec{u}]_E = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ | $-2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{u}$ ✓

Lemma: U a VS, $E = \{\vec{e}_1, \dots, \vec{e}_n\}$,

the fn $[\cdot]_E : U \to \mathbb{R}^n$ is linear.

$$\vec{u} \mapsto [\vec{u}]_E$$

Pf/ Let $\vec{u} = a_1\vec{e}_1 + \dots + a_n\vec{e}_n$, $\vec{v} = b_1\vec{e}_1 + \dots b_n\vec{e}_n$, $r \in \mathbb{R}$

$$[r\vec{u}]_E = [ra_1\vec{e}_1 + \dots + ra_n\vec{e}_n]_E = \begin{bmatrix} ra_1 \\ \vdots \\ ra_n \end{bmatrix}_E = r\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_E = r[u]_E$$

$$[\vec{u}+\vec{v}] = [(a_1+b_1)\vec{e}_1 + \dots + (a_n+b_n)\vec{e}_n]_E = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix}_E = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_E + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_E = [u]_E + [v]_E$$

$\vec{u} = 1 + y^2 - x^3 \qquad \vec{v} = 3 - x + 4x^3$

$\vec{u} + \vec{v} = 4 - x + y^2 + 3x^3$

$$[u]_E = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \qquad [v]_E = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix}$$

$$[u]_E + [v]_E = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix} = [\vec{u} + \vec{v}]_E$$

$$
\begin{tikzcd}
U \arrow[r, "L"] \arrow[d, "{[\cdot]_E}"] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "{[\cdot]_F}"] \\
\mathbb{R}^n \arrow[r, dashed, "A"] & \mathbb{R}^m
\end{tikzcd}
\qquad
\begin{tikzcd}
\vec{u} \arrow[r, mapsto, "L"] \arrow[d, mapsto, "{[\cdot]_E}"] \arrow[dr, phantom, "\circlearrowright"] & L(\vec{u}) \arrow[d, mapsto, "{[\cdot]_F}"] \\
{[\vec{u}]_E} \arrow[r, mapsto, dashed, "A"] & A[\vec{u}]_E = [L(\vec{u})]_F
\end{tikzcd}
$$

Claim: $\exists A$ s.t.

$$A[\vec{u}]_E = [L(\vec{u})]_F$$

Thm: U, V F.d. VS. $\quad L: U \to V$

$E = \{\vec{e}_1, \ldots, \vec{e}_n\}$ basis for U

$F = \{\vec{f}_1, \ldots, \vec{f}_m\}$ basis for V

Then $\exists\ A \in M_{m \times n}$ that represents L wrt E and F.

That is, $L\vec{u} = \vec{v}$ iff $A[\vec{u}]_E = [v]_F$.

$A = [\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n]$ where $\vec{c}_j = [L(\vec{e}_j)]_F$.

Let $F = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ basis for $\mathbb{R}^2$

$E = \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$ std basis for $\mathbb{R}^3$

$$L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y - z \\ x + y + z \end{bmatrix}$$

1) find matrix wrt std bases

$$L \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad L \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$L \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

2) now find wrt E, F

$$L \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{f}_1, \quad \left[L \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]_F = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{f}_2, \quad \left[L \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right]_F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{f}_2, \quad \left[L \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]_F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$${}_F A_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$S = \text{span}\{\overset{s_1}{e^x}, \overset{s_2}{xe^x}, \overset{s_3}{x^2e^x}\} \leq C([a,b], \mathbb{R})$$

$$\frac{d}{dx}: S \to S$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{d}{dx} e^x = e^x = \vec{s}_1 + 0\vec{s}_2 + 0\vec{s}_3$$

$$\frac{d}{dx} xe^x = e^x + xe^x = \vec{s}_1 + \vec{s}_2 + 0\vec{s}_3$$

$$\frac{d}{dx} x^2e^x = 2xe^x + x^2e^x = 2\vec{s}_2 + \vec{s}_3$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{d}{dx} \cdot 10e^x + 15xe^x - 3x^2e^x = 25e^x + 9xe^x - 3x^2e^x$$

$$A \begin{bmatrix} 10 \\ 15 \\ -3 \end{bmatrix} = \begin{bmatrix} 25 \\ 9 \\ -3 \end{bmatrix}$$

$$E = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ 2z \\ -x+y+z \end{bmatrix} \qquad {}_{std}A_{std} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$L \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\vec{f}_1 \qquad L \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\vec{f}_2 \qquad L \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\vec{f}_3$$

$${}_{std}A_E = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \qquad {}_F A_E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$