- $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$ ker, image

  $L(r\vec{x}) = rL(\vec{x})$

If $A \in M_{m \times n}$ matrix, then

$L(\vec{x}) = A\vec{x}$ is a LT $\mathbb{R}^n \to \mathbb{R}^m$

$\ker(L) = \text{null}(A)$

$\text{im}(L) = \text{col}(A)$

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "{[\cdot]_E}"] & V \arrow[d, "{[\cdot]_F}"] \\ \mathbb{R}^n \arrow[r, dashed, "\exists A"] & \mathbb{R}^n \end{tikzcd}$$

$L : P_3(x) \to P_3(x)$

$L(f) = (1+x)f'' - f'$

$E = \{1, x, x^2, x^3\}$

$L(1) = 0 \qquad L(x) = 0 - 1$

$L(x^2) = (1+x)2 - 2x = 2$

$L(x^3) = (1+x)6x - 3x^2 = 6x + 3x^2$

$$A = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_R$$

$\mathrm{rk}(A) = \dim(\mathrm{row}(A)) = 2 = \dim \mathrm{col}(A) = \dim(\mathrm{im}\, A)$

$\dim(\mathrm{im}(L)) = 2$

Basis for $\mathrm{col}(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \\ 0 \end{bmatrix} \right\}$ Basis for $\mathrm{im}(L) = \{-1, 6x + 3x^2\}$

$$A = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_R$$

$$\text{nullspace}(A) = \text{null}(A_R) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \,\middle|\, x_2 = 2x_3,\ x_4 = 0 \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{basis for null}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\ker(L) = \{ \alpha(1) + \beta(2x + x^2) \}, \quad \text{basis } \{1, 2x + x^2\}.$$

Rank-Nullity: $\boxed{rk(A)} + \boxed{nullity(A)} = \boxed{\#cols(A)}$

R-N for VS: $L: U \to V$ linear, then

$\dim(\mathrm{im}(L)) + \dim(\ker(L)) = \dim U$

Pf/ WTS $A[\vec{u}]_E = [L(\vec{u})]_F$ *

$$A = [\vec{c}_1 \cdots \vec{c}_n] = \big[[L(\vec{e}_1)]_F \cdots [L(\vec{e}_n)]_F\big]$$

Let $\vec{u} \in U$. Write $\vec{u} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$

$$[L(\vec{u})]_F = [a_1 L(\vec{e}_1) + a_2 L(\vec{e}_2) + \cdots + a_n L(\vec{e}_n)]_F$$

$$= a_1 [L(\vec{e}_1)]_F + \cdots + a_n [L(\vec{e}_n)]_F$$

$$A[\vec{u}]_E = A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \big[[L(\vec{e}_1)]_F \cdots [L(\vec{e}_n)]_F\big]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 [L(\vec{e}_1)]_F + \cdots + a_n [L(\vec{e}_n)]_F.$$

Know $\ker(L)$ is a subspace of U if $L: U \to V$.

Prop: Let V v.s., U ss of V.

Then $\exists\, L: V \to V$ s.t. $\ker(L) = U$.

Pf/ Let $\{\vec{u}_1, \dots, \vec{u}_n\}$ basis for U

$\{\vec{u}_1, \dots, u_n, \vec{v}_1, \dots, \vec{v}_m\}$ basis for V.

$$\boxed{\begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix}}$$

def $L(a_1\vec{u}_1 + \dots + a_n\vec{u}_n + b_1\vec{v}_1 + \dots b_m\vec{v}_m) = b_1\vec{v}_1 + \dots + b_m\vec{v}_m$.

$\ker = \operatorname{span}\{\vec{u}_1, \dots, \vec{u}_n\} = U$.

Any solution to a linear system is a SS

is $N(A)$

is $\ker(L)$

§4.4 Isomorphisms

Dfn: $f: U \to V$ a fn

if $\exists g: V \to U$ s.t.

$g(f(\vec{u})) = \vec{u} \ \forall \vec{u} \in U$,

and $f(g(\vec{v})) = \vec{v} \ \forall \vec{v} \in V$,

then $g = f^{-1}$ is an inverse of

f, and f is invertible.

If L is an invertible LT

we say L is an isomorphism

$L: U \xrightarrow{\sim} V$

U and V are isomorphic

write $U \cong V$, if

$\exists L: U \xrightarrow{\sim} V$

$P_z(x,y,z) = (x,y,0)$

$(3,1,2) \mapsto (3,1,0)$

$(3,1,?) \mapsfrom (3,1,0)$

not an iso!

$I_n\ (x,y,z) \mapsto (x,y,z)$

isomorphism

Ex: $f: U \to \mathbb{R}^n$

$\vec{u} \mapsto [u]_E$

is an iso

inverse $g: \mathbb{R}^n \to U$

$$g\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1\vec{e}_1 + \cdots + a_n\vec{e}_n$$

$$\begin{array}{ccc} U & \xrightarrow{L} & V \\ \big\downarrow [\cdot]_E & & \big\uparrow [\cdot]_F^{-1} \\ \mathbb{R}^n & \xrightarrow{\exists A} & \mathbb{R}^m \end{array}$$

$E = \{1, x, x^2, x^3\}$

then

$[\cdot]_E : P_3 \xrightarrow{\sim} \mathbb{R}^4$

so $P_3 \cong \mathbb{R}^4$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

$g : \mathbb{R}^2 \to \mathbb{R}^2$ by $g\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{bmatrix}$ Claim $g = f^{-1}$

$$g\left(f\begin{bmatrix} x \\ y \end{bmatrix}\right) = g\begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} \frac{x+y+x-y}{2} \\ \frac{x+y-x+y}{2} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Dfn: • f is one-to-one / injective / 1-1

if: when $f(\vec{x}) = f(\vec{y})$ then $\vec{x} = \vec{y}$.

•

• $f: U \to V$ is onto / surjective if

$\forall \ \vec{y} \in V, \ \exists \ \vec{x} \in U$ s.t. $f(\vec{x}) = \vec{y}$.

$P_2: \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$ onto

$P_2: \mathbb{R}^3 \to \mathbb{R}^3$

not onto $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

f is bijective if 1-1 <u>and</u> onto.

Prop: $L: U \to V$ linear. then

1) L is 1-1 iff $\ker L = \{\vec{0}\}$

2) L is invertible iff L is bijective.

Pf/ 1) HW

2) $L: U \to V$ bijection. Define $T: V \to U$ by:

Let $\vec{v} \in V$. By onto, $\exists\ \vec{u} \in U$ s.t. $L(\vec{u}) = \vec{v}$.

by 1-1, this $\vec{u}$ is unique. define $T(\vec{v}) = \vec{u}$. $T = L^{-1}$.

Conversely, suppose L is invertible.

Suppose $L(\vec{x}) = L(\vec{y})$. Then $\underbrace{L^{-1}(L(\vec{x}))}_{\vec{x}} = \underbrace{L^{-1}(L(y))}_{\vec{y}}$

Let $\vec{v} \in V$. then $L(L^{-1}(\vec{v})) = \vec{v}$ and $L^{-1}(\vec{v}) \in U$.

Cor: $L: U \to V$ linear. L is 1-1 iff
$\dim(U) = \dim(L(U))$
L is onto iff $\dim(U) - \dim(\ker L) = \dim V$
If $\dim U = \dim V$, L is iso iff ker is trivial.

Pf/ know $\dim U = \dim \ker + \dim L(U)$.

How do we find an inverse?

L a LT, A the matrix of L.

Then A^{-1} is the matrix of L^{-1}!

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ 2x+3y+2z \\ x+5y+4z \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 5 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/3 & -4/3 & 1/3 \end{array}\right]$$

$$L^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2/3\,a + 1/3\,b - 1/3\,c \\ -2a + b \\ 7/3\,a - 4/3\,b + 1/3\,c \end{bmatrix}$$

$$E : P_2 \to \mathbb{R}^3 \qquad E(f) = \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \end{bmatrix}$$

$$E(a_0 + a_1 x + a_2 x^2) = \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array} = A \quad \Big/ \quad A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}.$$

$$E^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = b(1) + \left(-\tfrac{1}{2}a + \tfrac{1}{2}c\right)x + \left(\tfrac{1}{2}a - b + \tfrac{1}{2}c\right)x^2.$$

$$E^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = b(1) + \left(-\frac{1}{2}a + \frac{1}{2}c\right)x + \left(\frac{1}{2}a - b + \frac{1}{2}c\right)x^2.$$

what if. $f(-1) = 3$, $f(0) = 1$, $f(1) = 2$

$$f(x) = 1 + \left(-\frac{1}{2}\right)x + (3/2)\,x^2$$