$\frac{d}{dx} : P_3(x) \to P_3(x)$ is this an isomorphism?

is it 1-1?

$\frac{d}{dx} 1 = 0 = \frac{d}{dx} 0$ not 1-1 | 1-1 iff $\ker = \{\vec{0}\}$

$1 \in \ker$

not onto either

$$\left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

$\frac{d}{dx} : \operatorname{span}\{x, x^2, x^3\} \to \operatorname{Span}\{1, x, x^2\}$

$\frac{d}{dy} : xP_2(x) \to P_2(x)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$a + bx + cx^2$

$\mapsto ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$

Prop: V a vs, then $V \cong V$

Pf/ $Id_V(\vec{x}) = \vec{x}$ is an iso

Prop: U, V vs, $E = \{\vec{e}_1, \dots, \vec{e}_n\}$ basis for U, $L: U \to V$

$L(E)$ spans $L(U)$. L is an iso iff

1) $L(E)$ is a basis for V

2) $\# L(E) = \# E$.

$L(\vec{x}) = 1 : \mathbb{R}^2 \to \mathbb{R}$ | $L\left(\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}\right) = \cancel{\{1, 1\}} = \{1\}$

Cor: U, V v.s. $\dim U = \dim V$.

Then $U \cong V$.

Pf/ Let $E = \{\vec{e}_1, \dots, \vec{e}_n\}$ basis for U

$F = \{\vec{f}_1, \dots, \vec{f}_n\}$ basis for V

def $L: U \to V$ by

$L(a_1\vec{e}_1 + \dots + a_n\vec{e}_n) = a_1\vec{f}_1 + \dots + a_n\vec{f}_n$ is an iso.

Cor: $L: V \to V$ iso iff sends a basis to a basis.

§5 Eigenvectors and Eigenvalues

§5.1 Eigenvectors

Dfn: $L: V \to V$ LT, $\lambda \in \mathbb{R}$

If $\exists\ \vec{v} \in V$ s.t. $L(\vec{v}) = \lambda \vec{v}$,

then λ is an eigenvalue of L,

and $\vec{v}$ is an eigenvector w/ eigenvalue λ.

$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is evec w/ eval 3

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ evec?

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$ no

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$

$R_{\pi/2} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

$\lambda x = -y$
$\lambda y = x$ $\Rightarrow$ $\lambda^2 y = -y$
$\lambda^2 = -1$ or $y = 0$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is only EV. Technically.

$V = D(\mathbb{R}, \mathbb{R})$, $\frac{d}{dx} : V \to V$

Let $f = e^{rx}$ then $\frac{d}{dx} f = \frac{d}{dx} e^{rx} = r e^{rx} = rf$

So f is evec w/ eival r.

$\frac{d}{dx} y = ry$

Prop: $L: V \to V$ LT. Then $\vec{x} = I_n \vec{x}$

$\vec{v}$ is evec w/ eval λ iff

$\vec{v} \in \ker(L - \lambda Id_V)$.

Pf/ $\vec{v}$ is evec iff $L(\vec{v}) = \lambda \vec{v} = \lambda Id_V \vec{v}$

iff $\vec{0} = L(\vec{v}) - \lambda(Id_V(\vec{v})) = (L - \lambda Id_V)(\vec{v})$

iff $\vec{v} \in \ker(L - \lambda Id_V)$

Cor: Let $E_\lambda = \{\vec{v} \in V \mid L(\vec{v}) = \lambda \vec{v}\}$ the <u>eigenspace</u> of L

corresponding to λ, is a SS of V.

Cor: $L: V \to V$ is invertible iff 0 not an eval of L.

Pf: $E_0 = \ker(L - 0I) = \ker(L)$

Prop: $L: V \to V$ linear. if $E = \{\vec{e}_1, \dots, \vec{e}_n\}$ is a set of evec w/ distinct eval, then E is LI.

Pf: Let λ_i be eval of $\vec{e}_i$.

Suppose $a_1\vec{e}_1 + \cdots + a_n\vec{e}_n = \vec{0}$.

Let k be smallest # so that $\vec{e}_1, \ldots, \vec{e}_k$ LD.

then can write $b_1\vec{e}_1 + \cdots + b_k\vec{e}_k = \vec{0}$ w/ $b_k \neq 0$

$$\vec{e}_k = \frac{-b_1}{b_k}\vec{e}_1 + \cdots + \frac{-b_{k-1}}{b_k}\vec{e}_{k-1}$$

$$L(\vec{e}_k) = \frac{-b_1}{b_k}L(\vec{e}_1) + \cdots + \frac{-b_{k-1}}{b_k}L(\vec{e}_{k-1})$$

$$\lambda_k\vec{e}_k = \frac{-b_1}{b_k}\lambda_1\vec{e}_1 + \cdots + \frac{-b_{k-1}}{b_k}\lambda_{k-1}\vec{e}_{k-1}$$

$$\left(\vec{e}_k = \frac{-b_1}{b_k}\vec{e}_1 + \cdots + \frac{-b_{k-1}}{b_k}\vec{e}_{k-1} \right) \lambda_k$$

$$\lambda_k \vec{e}_k = \frac{-b_1}{b_k}\lambda_1 \vec{e}_1 + \cdots + \frac{-b_{k-1}}{b_k}\lambda_{k-1}\vec{e}_{k-1}$$

$$\vec{0} = \frac{-b_1}{b_k}(\lambda_1 - \lambda_k)\vec{e}_1 + \cdots + \frac{-b_{k-1}}{b_k}(\lambda_{k-1} - \lambda_k)\vec{e}_{k-1}$$

$\vec{e}_1, \ldots, \vec{e}_{k-1}$ LI all coeffs are 0.

$\frac{b_i}{b_k}$ not all 0. so at least one $\lambda_i - \lambda_k = 0$. $\Rightarrow \Leftarrow$ distinct eval,

QED.

$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Find evec, eval, espace. $A\vec{x} = \lambda \vec{x}$

$A\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$ solve $\begin{bmatrix} 3x \\ 2y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in E_3$ $E_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in E_2$ $E_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$B = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} \qquad \begin{bmatrix} 7x + 2y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \qquad \begin{aligned} (7-\lambda)x + 2y &= 0 \\ 3x \quad (8-\lambda)y &= 0 \end{aligned}$$

$$\begin{bmatrix} 7-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} \to \begin{bmatrix} 3 & 8-\lambda \\ 0 & 2 + (8-\lambda)(\lambda-7)/3 \end{bmatrix}$$

$$\to \begin{bmatrix} 3 & 8-\lambda \\ 0 & -\lambda^2 + 15\lambda - 50 \end{bmatrix}$$

$$-\lambda^2 + 15\lambda - 50 = 0$$

Solve $-\lambda^2 + 15\lambda - 50 = 0$

$$\Downarrow$$

$$(\lambda - 5)(\lambda - 10) = 0$$

$\lambda = 5, 10$

$B = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}, \quad \lambda \in \{5, 10\}$

$\begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix} \Rightarrow$

$(7-\lambda) x + 2y = 0$
$3x \quad (8-\lambda) = 0,$

$\lambda = 5 \Rightarrow$ $2x + 2y = 0$, $3x + 3y = 0$ $\Rightarrow x = -y$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in E_5$

$\lambda = 10 \Rightarrow$ $-3x + 2y = 0$, $3x - 2y = 0$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in E_{10}$

§5.2 Determinants

Dfn: $A \in M_{n \times n}$ w/ n distinct evals

Then $\det(A)$ is the product of the evals

$$\det \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = 3 \cdot 2 = 6$$

$$\det \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} = 5 \cdot 10 = 50$$

if not distinct evals:

Roughly: product of evals "up to multiplicity"

Formally:

$$\det A = \prod_{\lambda} \lambda^{e_\lambda} \quad \text{where } e_\lambda = \dim \ker (A - \lambda I)^n.$$