Determinants

det A "=" product of eigenvalues
("up to multiplicity")

$A\vec{x} = \lambda\vec{x}$

$N(A\vec{x} - \lambda\vec{x})$

Laplace Formula

Dfn: $A = (a_{ij}) \in M_{n\times n}$

The i,j minor matrix of A is a $(n-1)\times(n-1)$ matrix M_{ij}:
delete i^{th} row, j^{th} column

The i,j minor is $\det M_{ij}$

The i,j cofactor is $(-1)^{i+j}\det(M_{ij})$

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix}$

$M_{11} = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 3 \end{bmatrix}$

minor $= \det \begin{bmatrix} -2 & -1 \\ 3 & 3 \end{bmatrix} = -3$

Cofactor $= -1^{1+1}(-3) = -3$

$M_{3,2} = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}$

minor $= \det \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} = -13$

Cofactor $= (-1)^{3+2}(-13) = 13$

Fact (Cofactor expansion): Let $A \in M_{n\times n}$

- if $A \in M_{1\times 1}$, $\det[a_{11}] = a_{11}$

- else

$$\det(A) = \sum_{i=1}^{n} a_{ki} A_{ki} = a_{k1}A_{k1} + \cdots + a_{kn}A_{kn}$$

$$= \sum_{i=1}^{n} a_{ik} A_{ik} = a_{1k}A_{1k} + \cdots + a_{nk}A_{nk}$$

for any k.

Ex: $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$\det A = \cancel{0(-1)^{3+1}\det\begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}} + \cancel{0(-1)^{3+2}\det\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}}$$

$$+ 2(-1)^{3+3}\det\begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix}$$

$$= 2\left(\cancel{0(-1)^{2+1}\det[2]} + 5(-1)^{2+2}\det[3]\right)$$

$$= 2(5 \cdot 1 \cdot 3) = 30$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix}$$

$$\det A = 3(-1)^{3+1} \det \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} + 3(-1)^{3+2} \det \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} + 3(-1)^{3+3} \det \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix}$$

$$= 3\left(1(-1)^{1+1} \det[-1] + 2(-1)^{1+2} \det[-2]\right)$$

$$-3\left(3(-1)^{1+1} \det[-1] + 5(-1)^{2+1} \det[2]\right)$$

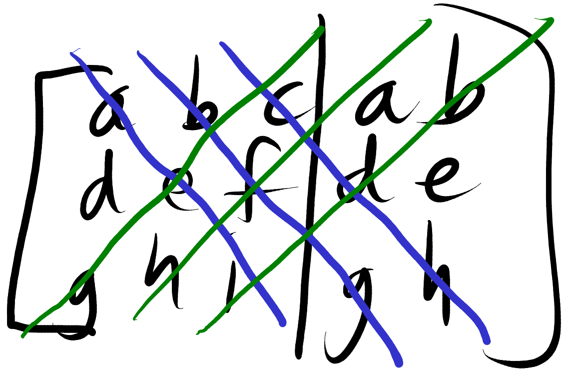
$$+3\left(1(-1)^{1+2} \det[5] - 2(-1)^{2+2} \det[3]\right)$$

$$= 3(-1+4) - 3(-3-10) + 3(-5-6) = 9 + 39 - 33 = 15.$$

Prop

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a(-1)^{1+1}\det[d] + b(-1)^{1+2}\det[c]$$
$$= ad - bc$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - gec - hfa - idb$$



Properties of det

Prop: if A is triangular, then $\det A$ is product of diagonal

PF/ cofactor expansion.

Prop: if A has a row or column of all zeroes then $\det A = 0$

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 9 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Prop: $\det A^T = \det A$

PF/ expand along 1^{st} row of A, 1^{st} col of A^T

Fact (row operations)

1) Interchanging 2 rows multiplies det by -1.

2) Multiplying a row by a scalar r multiplies det by r.

(Cor: $\det(rA) = r^n \det A$)

3) adding mult of one row to another: does nothing

4) $$\det\begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{r}_n \end{bmatrix} + \det\begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{b}_i \\ \vdots \\ \vec{r}_n \end{bmatrix} = \det\begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{a}_i + \vec{b}_i \\ \vdots \\ \vec{r}_n \end{bmatrix}$$

Ex: $\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 1 \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = (1 \cdot 2) - (1 \cdot 1) = 1.$

$\det \begin{bmatrix} 3 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 3 \det(\quad) = 3.$

$\det \begin{bmatrix} 4 & 4 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 1 + 3 = 4$

$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1.$

Cor: $\det A = 0$ iff rows are LD.

Prop: A matrix A is invertible iff $\det A \neq 0$.

Pf/ 1) Evals: $\det A = 0$ iff 0 is an eval.

iff E_0 is nontrivial

iff $\ker(A - 0I) = \ker(A)$ is nontrivial.

2) cofactor: A is invertible iff row-eq to I_n

know $\det(I_n) = 1 \neq 0$, so $\det(A) \neq 0$.

If A not invertible, it's row-eq to a matrix w/ row of zeroes

Fact: $\det(AB) = \det(A) \cdot \det(B)$.

Pf/ ugly induction on cofactor expansion

Cor: if $\det A \neq 0$, i.e. if A is non-singular,

then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Fact: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

§5.3 Characteristic Polynomials

Dfn: $\chi_A(\lambda) = \det(A - \lambda I)$ is characteristic polynomial of A.

Prop: λ is an eval of A iff $\chi_A(\lambda) = 0$.

Pf/ $\vec{v}$ is an eigenvector of A iff $A\vec{v} = \lambda\vec{v}$ for some λ

iff $\vec{v} \in \ker(A - \lambda I)$ for some λ.

So λ is an eval iff $\ker(A - \lambda I)$ is nontrivial

iff $\det(A - \lambda I) = 0$

iff $\chi_A(\lambda) = 0$.

Ex! $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2-0 \\ 3-0 & -2-\lambda \end{bmatrix}$

$$\chi_A(\lambda) = \det \begin{bmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix} = (3-\lambda)(-2-\lambda) - (6)$$
$$= \lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3)$$

So evals are $4, -3$

$\ker(A - 4I) = \ker \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} = \ker \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, Basis for E_4 is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\ker(A + 3I) = \ker \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} = \ker \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$

basis for E_{-3} is $\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ $\quad A \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$$

$$\det \begin{bmatrix} 5-\lambda & 1 \\ 3 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda) - 3 = \lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2)$$

$$E_2 = \ker \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = \ker \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\} \quad \text{Basis: } \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

$$E_6 = \ker \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} = \ker \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \text{Basis} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$\chi_A(\lambda) = \det \begin{bmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{bmatrix} = (2-\lambda)(-2-\lambda)(2-\lambda) + (-3) + (-3) - \Big((-2-\lambda) + (-3)(2-\lambda) + (-3)(2-\lambda)\Big)$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda^2 - 2\lambda + 1) = -\lambda(\lambda-1)^2$$

0 is an eval, 1 is an eval "twice"

$$E_0 = \ker(A) = \ker \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ Basis for E_0 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$E_1 = \ker(A - I) = \ker \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left\{ \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ $B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Prop: If $A \in M_{n \times n}$ and n is odd,
then A has an eval.

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \chi_A(\lambda) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$

no evals

(Prop: if we work over complex numbers $\mathbb{C}$,
every matrix has an eval.)

Ex! $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ evals are 2, 4, 2

$$E_4 = \ker \underbrace{\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}}_{B-4I} = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{Basis } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E_2 = \ker \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B-2I} = \left\{ \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{Basis } \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$