$$A\vec{x} = \lambda\vec{x}$$

$$A(\vec{x}_1+\vec{x}_2+\vec{x}_3) = \lambda_1\vec{x}_1 + \lambda_2\vec{x}_2 + \lambda_3\vec{x}_3$$

$V \to V$ endomorphism

$V \xrightarrow{\sim} V$ automorphism

§6 Similarity and change of basis

§6.1 Change of basis

Dfn: $L : V \to V$ iso iff it sends a basis to a basis.

Then we call it a <u>change of basis map</u>

The matrix is a <u>transition matrix</u>

Ex: $E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad F = \left\{ \overset{f_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{f_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{f_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$

Want $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L^{-1}\Bigg(\underset{f_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}\Bigg) = \underset{f_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}},$$

$$L^{-1}\Bigg(\overset{f_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}\Bigg) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underset{f_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} - \underset{f_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$L^{-1}\Bigg(\overset{f_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\Bigg) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underset{f_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} - \underset{f_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

Suppose $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Want $[\vec{u}]_F$

$$x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{Then } [\vec{u}]_F = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The matrix A: sends elements of E to elements of F

A^{-1} sends coordinates in E to coordinates in F

The matrix A: sends elements of E to elements of F

A^{-1} sends coordinates in E to coordinates in F

A sends coordinates in F to coordinates in E

A: Transition matrix from F to E

A^{-1} Transition matrix from E to F.

"contravariant"

Suppose $\vec{v} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\\0\end{bmatrix} + 5\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad [\vec{v}]_F = \begin{bmatrix}2\\3\\5\end{bmatrix}$

$$[\vec{v}]_E = A[\vec{v}]_F = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}2\\3\\5\end{bmatrix} = \begin{bmatrix}10\\8\\5\end{bmatrix}$$

$\vec{u} = 5\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 7\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\quad [u]_E = \begin{bmatrix}5\\2\\7\end{bmatrix}$

$$[\vec{u}]_F = A^{-1}[\vec{u}]_E = \begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}5\\2\\7\end{bmatrix} = \begin{bmatrix}3\\-5\\7\end{bmatrix}$$

$$3\begin{bmatrix}1\\0\\0\end{bmatrix} - 5\begin{bmatrix}1\\1\\0\end{bmatrix} + 7\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\2\\7\end{bmatrix}.$$

EX: $P_2(x)$ $\quad E=\{1, x, x^2\}$ $\quad F=\{1, 2x, 4x^2-2\}$

Want $[a+bx+cx^2]_F$

$$A=\begin{bmatrix}1 & 0 & -2\\ 0 & 2 & 0\\ 0 & 0 & 4\end{bmatrix} \qquad \text{Want } A^{-1}=\begin{bmatrix}1 & 0 & 1/2\\ 0 & 1/2 & 0\\ 0 & 0 & 1/4\end{bmatrix}$$

transition $F \to E$ $\qquad$ transition map $F \to E$

$$[a+bx+cx^2]_F=\begin{bmatrix}1 & 0 & 1/2\\ 0 & 1/2 & 0\\ 0 & 0 & 1/4\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}a+c/2\\ b/2\\ c/4\end{bmatrix}$$

$$a+bx+cx^2 = a+\frac{c}{2}+\frac{b}{2}(2x)+\frac{c}{4}(4x-2).$$

Can paste together

$$F = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \qquad G = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$F \to E$

Want $B^{-1}A$

$F \to G$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$G \to E$

$$B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \qquad B^{-1}A = \frac{1}{2} \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}$$

$$F = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \quad G = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$B^{-1}A = \frac{1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix} \quad [\vec{u}]_G = B^{-1}[\vec{u}]_E = B^{-1}A[\vec{u}]_F$$

$F \to G$

$$\frac{1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} ? & ? & 11 \\ & & 7 \end{bmatrix}$$

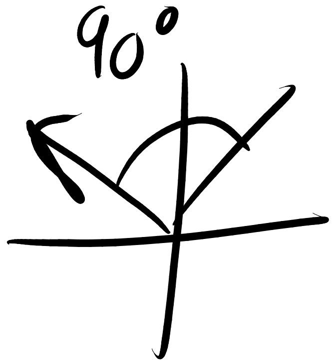
§ 6.2 Similarity

$$\begin{tikzcd} L \arrow[r, "E"] \arrow[dr, "F"'] & A \\ & B \end{tikzcd}$$

$R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$

std basis $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$F = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$



$R_{\frac{\pi}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$R_{\frac{\pi}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$$R_{\frac{\pi}{2}} \xrightarrow{E} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A$$

$$R_{\frac{\pi}{2}} \xrightarrow{F} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = B$$

$$\underset{F \to E}{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\underset{E \to F}{U^{-1}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A[\vec{u}]_E = U B [\vec{u}]_F$$

$$= U B U^{-1} [\vec{u}]_E$$

Thus $A = UBU^{-1}$

$$B = U^{-1} A U$$

$$U^{-1}AU = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = B$$

Dfn: $A, B \in M_{n\times n}$ are <u>similar</u> if $\exists U \in M_{n\times n}$ s.t. $U^{-1}AU = B$. Write $A \sim B$

Prop: $E = \{\vec{e}_1, \dots, \vec{e}_n\}$, $F = \{\vec{f}_1, \dots, \vec{f}_n\}$ bases for V

$L: V \to V$ linear $\quad U$ transition matrix $F \to E$.

If A reps L wrt E, B reps L wrt F

then $B = U^{-1}AU$.

$$\frac{d}{dx}: P_2(x) \to P_2(x) \quad E = \{1, x, x^2\}, \; F = \{1, 2x, 4x^2 - 2\}$$

$$A_E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_F = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{F \to E}{U} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \underset{E \to F}{U^{-1}} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$B_F = U^{-1} A_E U = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

Ex: $L: \mathbb{R}^3 \to \mathbb{R}^3$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+3y+z \\ 2x-y+3z \\ y-z \end{bmatrix}$$

$$F = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

$$A_E = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$B_F = U^{-1} A_E U = \begin{bmatrix} -25 & -16 & -4 \\ 49 & 31 & 7 \\ -38 & -25 & -7 \end{bmatrix}$$

$$\underset{F \to E}{U} = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \underset{E \to F}{U^{-1}} = \begin{bmatrix} -1 & -1 & 3 \\ 2 & 2 & -5 \\ -1 & -2 & 3 \end{bmatrix}$$

$$F \xleftarrow{U^{-1}} E \xleftarrow{A_E} E \xleftarrow{U} F$$

Prop: $A, B \in M_{n\times n}$, $A \sim B$. Then

- $rk(A) = rk(B)$
- $nullity(A) = nullity(B)$
- A invertible iff B invertible

- $\det(A) = \det(B)$
- $\chi_A(\lambda) = \chi_B(\lambda)$
- Set of eigenvalues are the same.

$col(A) \neq col(B)$

$row(A) \neq row(B)$

$N(A) \neq N(B)$

eigenvectors not the same.