§6.3 Determinant and trace.

Prop: $A, B \in M_n$, $A \sim B$. ($\exists\, U$ s.t. $B = U^{-1}AU$). Then

- $\det(A) = \det(B)$
- $\chi_A(\lambda) = \chi_B(\lambda)$
- A, B have same eigenvalues.

Pf/ • $\det(B) = \det(U^{-1}AU) = \det(U^{-1})\det(A)\det(U)$

$$= \frac{1}{\det(U)}\det(U)\det(A) = \det(A).$$

• $\chi_B(\lambda) = \det(B - \lambda I) = \det(A - \lambda I) = \chi_A(\lambda)$.

$U^{-1}(B - \lambda I)U = U^{-1}BU - U^{-1}\lambda I U = A - \lambda U^{-1}U = A - \lambda I$

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = U^{-1}AU = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

$\det(A) = 2$

$\chi_A(\lambda) = (2-\lambda)(1-\lambda)(1-\lambda)$

evals $= 2, 1, 1$

$2 - 5\lambda + 4\lambda^2 - \lambda^3$

$$\chi_B(\lambda) = \det \begin{bmatrix} -\lambda & 2 & 0 \\ 2 & 3-\lambda & 3 \\ 1 & -2 & 1-\lambda \end{bmatrix}$$

$$= \left(-\lambda(3-\lambda)(1-\lambda) + 6\lambda\right) - 2\left(2(1-\lambda) - 3\lambda\right)$$

$$= 2 - 5\lambda + 4\lambda^2 - \lambda^3$$

$\det(B) = \chi_B(0) = 2.$

Converse: not true!

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\chi_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$

$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\chi_B(\lambda) = \cancel{0}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\chi_I(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$

$$U^{-1} I U = U^{-1} U = I$$

So $A \not\sim I$

$\det(A) = \chi_A(0) = $ constant term of $\chi_A(\lambda)$

other important coeff $\quad \lambda^{n-1}$ coefficient.

Dfn: $L: V \to V$ a LT

define trace of L to be

$Tr(L) = (-1)^{n-1} a_{n-1}$ where $\chi_L(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0$.

If $A \in M_n$, $Tr(A) = (-1)^{n-1} a_{n-1}$ where

$\chi_A(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0$.

Prop: • $A \sim B$. Then $Tr(A) = Tr(B)$

• $Tr(A)$ is sum of eigenvalues.

$\chi_A(x) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$, then

$a_{n-1} = (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

$Tr(A) = (-1)^{n-1} a_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

• $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$ is the sum of the diagonal.

Ex:

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 1 \\ 2 & -3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 & 1 \\ 4 & -3 & -3 \\ 2 & 1 & 0 \end{bmatrix}$$

$Tr(A) = 3 + 4 + 2 = 9$ $\qquad Tr(B) = 5 - 3 + 0 = 2$ $\qquad A \not\sim B$

$$C = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 1 & 7 \\ 1 & 1 & 4 \end{bmatrix}$$ $Tr(C) = 4 + 1 + 4 = 9$ $\qquad C \not\sim B$ $\qquad C \overset{?}{\sim} A$

Prop: Trace is linear.

- $Tr(A+B) = Tr(A) + Tr(B)$
- $Tr(rA) = r\,Tr(A)$
- $Tr(A^T) = Tr(A)$.

$$A^T = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & -3 \\ 5 & 1 & 2 \end{bmatrix}$$

§ 6.4 Diagonalization

Dfn: $D \in M_n$ is <u>diagonal</u> if $a_{ij} = 0$ when $i \neq j$ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Prop: $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ a diagonal $n \times n$ matrix. Then:

- $D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & 0 \\ \vdots & 0 & \ddots & d_n^k \end{bmatrix}$
- D has evecs are $\vec{e}_i$ w/ evals d_i.
- $\mathbb{R}^n$ is spanned by evecs.
- $\det(D) = d_1 \cdots d_n$
  $= \prod_{i=1}^{n} d_i$

Dfn: a linear transformation L is diagonalizable if its matrix in some basis is diagonal.

- a matrix is diagonalizable if it's similar to a diagonal matrix.

Prop: $A \in M_n$. Then

- A is diagonalizable iff evecs span $\mathbb{R}^n$
- A is diagonalizable iff it has n LI eigenvectors.
- If A has n distinct evals, then A is diagonalizable

Pf/. Suppose A is diagonalizable. then $\exists\, U, D$ s.t. $A = U^{-1}DU$.

so let $\vec{f}_i = U^{-1}\vec{e}_i$. Then $F = \{\vec{f}_1, \dots, \vec{f}_n\}$ is a basis.

$$A\vec{f}_i = U^{-1}DUU^{-1}\vec{e}_i = U^{-1}D\vec{e}_i = U^{-1}d_i\vec{e}_i = d_i U^{-1}\vec{e}_i = d_i\vec{f}_i.$$

Conversely, suppose evecs of A span $\mathbb{R}^n$.

Then basis $F = \{\vec{f}_1, \dots, \vec{f}_n\}$ of evecs.

Let U be transition matrix $F \to E$

$$U^{-1}AU\vec{e}_i = U^{-1}A\vec{f}_i = U^{-1}\lambda_i\vec{f}_i = \lambda_i\vec{e}_i$$

$$U^{-1}AU = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

2) If A diagonalizable, there's a basis of evecs, so
n LI evecs.

If A has n LI evecs, that's a basis, and spans.

3) If A has n distinct evals
then the corresponding evecs are LI.

Then by 2, A is diag.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not have 2 distinct evals.
is diagonalizable. And diagonal.

Ex: $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ $E_4 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$E_{-3} = \text{span}\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$

U: change of basis $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\} \to$ std basis

$U = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ $U^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ $U' = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$

$U^{-1}AU = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 4 & -9 \end{bmatrix}$

$= \frac{1}{7}\begin{bmatrix} 28 & 0 \\ 0 & -21 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}.$ $U'^{-1}AU' = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$

$$A=\begin{bmatrix}2&-3&1\\1&-2&1\\1&-3&2\end{bmatrix}\qquad E_0=\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\qquad E_1=\operatorname{span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$

$$U=\begin{array}{c}\begin{bmatrix}1&3&-1\\1&1&0\\1&0&1\end{bmatrix}\\ \begin{array}{ccc}0&1&1\end{array}\end{array}\qquad U^{-1}=\begin{bmatrix}-1&3&-1\\1&-2&1\\1&-3&2\end{bmatrix}$$

$$U^{-1}AU=\begin{bmatrix}-1&3&-1\\1&-2&1\\1&-3&2\end{bmatrix}\begin{bmatrix}2&-3&1\\1&-2&1\\1&-3&2\end{bmatrix}\begin{bmatrix}1&3&-1\\1&1&0\\1&0&1\end{bmatrix}$$

$$=\begin{bmatrix}-1&3&-1\\1&-2&1\\1&-3&2\end{bmatrix}\begin{bmatrix}0&3&-1\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Ex: $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ $E_2 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, $E_4 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

$B \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ Jordan Canonical Form

$$\det A = \prod_\lambda \lambda^{e_\lambda} \qquad \text{where} \quad e_\lambda = \dim \ker(A - \lambda I)^n.$$

Generalized eigenvectors are $\ker(A - \lambda I)^n$

Prop: If $A = U^{-1}BU$, then

$$A^n = (U^{-1}BU)^n = U^{-1}B\cancel{U}\cancel{U^{-1}}B\cancel{U} \cdots \cancel{U^{-1}}BU = U^{-1}B^nU$$

Let $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$. Find A^5, A^{50}.

If $U^{-1}AU = D$, then $UDU^{-1} = UU^{-1}AUU^{-1} = A$.

$$A^5 = UD^5U^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}^5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1024 & 0 \\ 0 & -243 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 843 & 362 \\ 543 & -62 \end{bmatrix}. \quad A^{50} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4^{50} & 0 \\ 0 & -3^{50} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.$$

$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \sim D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Find A^n

$$A^n = UD^nU^{-1} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} = UDU^{-1} = A.$$

A is <u>idempotent</u>.

Cor: If A is diagonalizable and evals are 0 or 1, then $A^n = A$ for any n.