$D = U^{-1} A U$ where U is {evecs} $\to$ {std basis} $\quad U = [\vec{v}_1, \vec{v}_2 \dots \vec{v}_n]$

$A = U D U^{-1}$ so $A^n = U D^n U^{-1}$

Cor: if A diagonalizable, evals are 0 or 1, then $A^n = A$ $\forall n$.

$y' = ky$

Compute $e^A = \exp(A)$ $\qquad$ want $e^{\cdot}$ $\qquad \vec{y}' = A\vec{y}$

$e^A = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \dots \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$

$= I + UDU^{-1} + \frac{1}{2}(UDU^{-1})^2 + \frac{1}{3!}(UDU^{-1})^3 + \dots$

$= I + UDU^{-1} + \frac{1}{2} U D^2 U^{-1} + \frac{1}{3!} U D^3 U^{-1} + \dots$

$$e^D = \begin{bmatrix} e^{d_1} & & 0 \\ & e^{d_2} & \\ 0 & & \ddots \; e^{d_n} \end{bmatrix}$$

$= U\left(I + D + \frac{1}{2} D^2 + \frac{1}{3!} D^3 + \dots\right) U^{-1} = U e^D U^{-1}$

§6.5 Application: Markov Chains

Ex: suppose 70% live in suburbs, 30% live in cities.

each year: 6% suburbs → city

2% city → suburbs

$s(1) = .94\, s(0) + .02\, c(0) = .94 \cdot .7 + .02 \cdot .3 = 66.4\%$

$c(1) = .06\, s(0) + .98\, c(0) = .06 \cdot .7 + .98 \cdot .02 = 33.6\%$

$$\begin{bmatrix} s(n+1) \\ c(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix}}_{A} \begin{bmatrix} s(n) \\ c(n) \end{bmatrix}.$$

1 year $A \begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .664 \\ .336 \end{bmatrix}$

5 years $A^5 \begin{bmatrix} .7 \\ .3 \end{bmatrix}$

$$A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \qquad U^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$A = UDU^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .92 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$A^5 = UD^5U^{-1} \approx \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .66 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \approx \begin{bmatrix} .744 & .085 \\ .256 & .915 \end{bmatrix}$$

$$A^5\begin{bmatrix} .7 \\ .3 \end{bmatrix} \approx \begin{bmatrix} .55 \\ .45 \end{bmatrix}.$$

$$A^{10}\begin{bmatrix} .7 \\ .3 \end{bmatrix} = UD^{10}U^{-1}\begin{bmatrix} .7 \\ .3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .45 \\ .55 \end{bmatrix}$$

$$A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \qquad U^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

What is equilibrium? (steady state) (infinite limit)

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} UD^nU^{-1} = U\left(\lim_{n\to\infty} D^n\right)U^{-1}$$

$$= U\left(\lim_{n\to\infty}\begin{bmatrix} 1^n & 0 \\ 0 & .92^n \end{bmatrix}\right)U^{-1} = U\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}U^{-1} = \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix}$$

$$A^n\begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} 1/4(.7) + 1/4(.3) \\ \frac{3}{4}(.7) + 3/4(.3) \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}.$$

A Markov process

$\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}, \ldots$

Markov Chain

Transition matrix

$$A \quad \begin{array}{cc} \begin{matrix} s_1 & s_2 & s_3 & s_4 \end{matrix} & \\ \begin{bmatrix} | & | & | & | \\ | & | & | & \bullet \\ | & | & | & | \\ | & | & | & | \end{bmatrix} & \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{matrix} \end{array}$$

$$B = \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad U^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lim_{n\to\infty} B^n = U \lim_{n\to\infty} D^n U^{-1} = U \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U^{-1} = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}$$

$$\lim_{n\to\infty} B^n \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} .4v_1 + .4v_2 \\ .6v_1 + .6v_2 \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$$

$$\boxed{C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Equilibrium state: evec w/ eval 1

Guaranteed to converge <u>if</u>

1) all entries are positive. Or

2) only 1 evec w/ eval 1

Car dealership

$$C = \begin{array}{cccc} \text{sedans} & \text{sports} & \text{minivans} & \text{SUVs} \\ \end{array}$$

$$C = \begin{bmatrix} .80 & .10 & .05 & .05 \\ .10 & .80 & .05 & .05 \\ .05 & .05 & .80 & .10 \\ .05 & .05 & .10 & .80 \end{bmatrix}$$

$$\begin{bmatrix} .7 \\ .2 \\ .05 \\ .05 \end{bmatrix}$$

evecs $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$1 \quad .8 \quad .7 \quad .7$

Steady-state $= \begin{bmatrix} .25 \\ .25 \\ .25 \\ .25 \end{bmatrix}$

$A\vec{u} = \vec{u} = 1 \cdot \vec{u}$

§7 Inner Product Spaces and Geometry

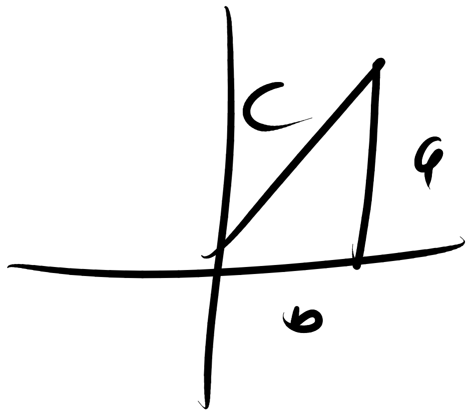
§7.1 The Dot Product

Dfn: $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. The <u>dot product</u> or <u>scalar product</u> is

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \in \mathbb{R}$$

Dfn: the <u>magnitude</u> of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2} = \sqrt{\vec{v} \cdot \vec{v}}.$$



$d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

Prop: $\vec{u},\vec{v} \in \mathbb{R}^n$. Then

1) Positive Definite: $\vec{u}\cdot\vec{u} \geq 0$, and if $\vec{u}\cdot\vec{u}=0$, then $\vec{u}=\vec{0}$

Pf/ $\vec{u}\cdot\vec{u} = u_1^2 + u_2^2 + \cdots u_n^2$ is a sum of squares.

so $\vec{u}\cdot\vec{u} \geq 0$. If $\vec{u}\cdot\vec{u}=0$, then $u_1^2 + \cdots u_n^2 = 0$

so $u_1^2 = \cdots = u_n^2 = 0$.

2) Symmetric $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$.

3) Bilinear: fix $\vec{u} \in \mathbb{R}$. Then the fns $L(\vec{x}) = \vec{x}\cdot\vec{u}$, $T(\vec{y}) = \vec{u}\cdot\vec{y}$ are linear

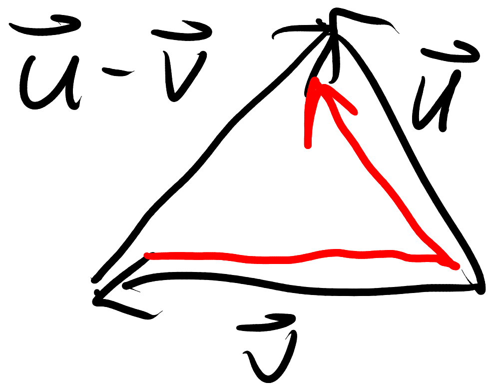
$\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} = \vec{u}\cdot(\vec{v}+\vec{w})$, $\vec{u}\cdot r\vec{v} = r(\vec{u}\cdot\vec{v})$

Pf/ $T(\vec{y})) = \vec{u}\cdot\vec{y} = \vec{u}^T\vec{y}$ is mult by a matrix.

Prop/ $\vec{u}, \vec{v} \in \mathbb{R}^n$ nonzero. $\angle$ btwn them is θ.

then $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$. thus $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$

Pf/ we get a Δ $\vec{u}, \vec{v}$



Law of Cosines

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

$$(\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

$$\cancel{\vec{u} \cdot \vec{u}} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \cancel{\vec{v} \cdot \vec{v}} = \cancel{\vec{u} \cdot \vec{u}} + \cancel{\vec{v} \cdot \vec{v}} - 2\|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$\cancel{-2}\, \vec{u} \cdot \vec{v} = \cancel{-2} \|\vec{u}\| \|\vec{v}\| \cos\theta.$$

QED.

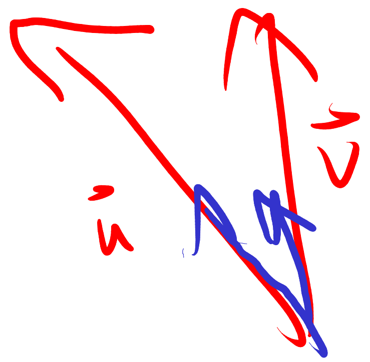
ex: $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

$\vec{u} \cdot \vec{v} = (3)(-1) + (4)(7) = 25$

$\|\vec{u}\| = 5 \quad \|\vec{v}\| = 5\sqrt{2}$

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \dfrac{25}{5\cdot 5\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\theta = \pi/4$.



Unit vectors

$\vec{u}$ is a <u>unit vector</u> if $\|\vec{u}\| = 1$.

if $\vec{v} \neq \vec{0}$, then

$\dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector.

unit vector of $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

unit vector of $\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix} -1 \\ 7 \end{bmatrix}$

$= \begin{bmatrix} -1/5\sqrt{2} \\ 7/5\sqrt{2} \end{bmatrix}$

Thm: Cauchy - Schwarz Inequality

$\vec{u}, \vec{v} \in \mathbb{R}^n$. Then

$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

Further, these are $=$ iff $\vec{u}$ and $\vec{v}$ have same direction or if $\vec{u}$ or $\vec{v} = \vec{0}$.

Pf/ $|\vec{u} \cdot \vec{v}| = |\|\vec{u}\| \|\vec{v}\| \cos\theta| = \|\vec{u}\| \|\vec{v}\| |\cos\theta| \leq \|\vec{u}\| \|\vec{v}\|$

since $|\cos\theta| \leq 1$.

$\cos\theta = \pm 1$ iff $\vec{u}, \vec{v}$ have same/opposite direction

$\vec{u}, \vec{v}$ same direction: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\|$

$\vec{u}, \vec{v}$ opposite dir: $\vec{u} \cdot \vec{v} = -\|\vec{u}\| \, \|\vec{v}\|$

$\vec{u}, \vec{v}$ 90° angle, then $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\pi/2) = 0.$

Dfn: we say $\vec{u}, \vec{v}$ are <u>orthogonal</u> or <u>pependicular</u>

if $\vec{u} \cdot \vec{v} = 0.$