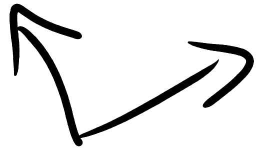
Dot product on $\mathbb{R}^n$

$\vec{u}\cdot\vec{v} = u_1v_1 + \cdots + u_nv_n$

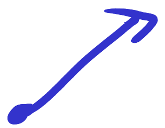
$|\vec{u}\cdot\vec{v}| = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

$\vec{u}\cdot\vec{v}$ maximized when they point in same direction



$\vec{u}\cdot\vec{v} = 0$ if $\vec{u}, \vec{v}$ are perpendicular/ <u>orthogonal</u>.

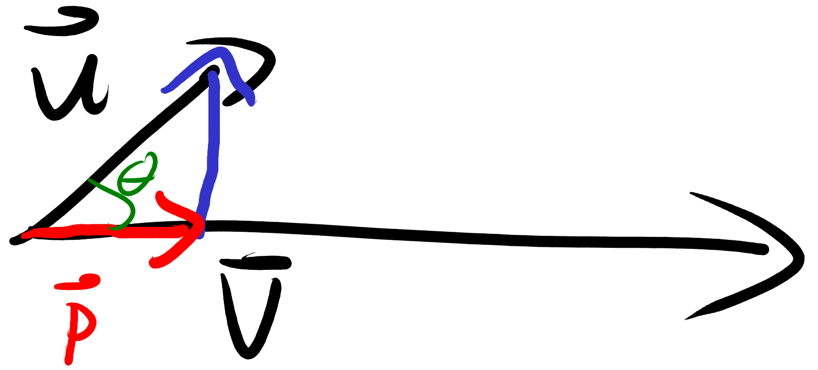
Ex. 1) $\vec{0}$ is orthogonal to <u>any</u> vector.



2) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 6 \end{bmatrix} = -12+12 = 0$ so orthogonal

3) $\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ $\quad 2\cdot a + 3b + 2c = 0$ $\quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ -5/2 \end{bmatrix}$

Dot product and orthogonality $\Rightarrow$ decomposition



$\vec{v}$ = Your preferences

$\vec{u}$ = the movie

$\vec{u} \cdot \vec{v}$ "=" how much you'll like the movie.

magnitude: $\|\vec{u}\| \cdot \cos\theta = \|\vec{p}\|$

$$\cancel{\|\vec{u}\|} \frac{\vec{u} \cdot \vec{v}}{\cancel{\|\vec{u}\|}\, \|\vec{v}\|} = \|\vec{p}\|$$

find red arrow.

direction: $\dfrac{\vec{V}}{\|\vec{V}\|}$

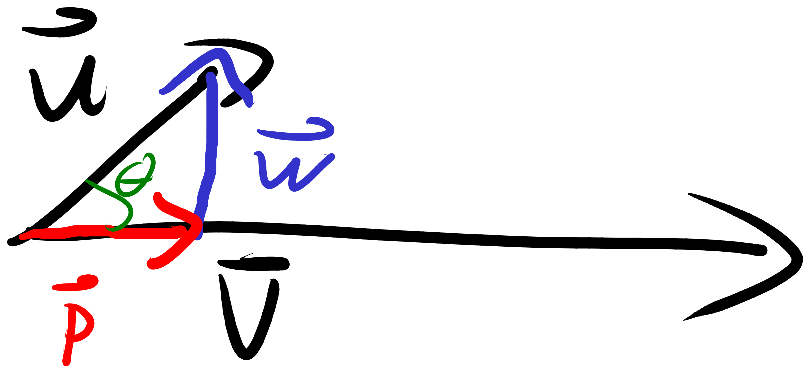
$$\|\vec{p}\| = \frac{\vec{u} \cdot \vec{V}}{\|\vec{v}\|}$$

$$\vec{p} = \frac{\vec{u} \cdot \vec{V}}{\|\vec{v}\|} \frac{\vec{V}}{\|\vec{v}\|} = \frac{\vec{u} \cdot \vec{V}}{\|\vec{v}\|^2} \vec{v} = \boxed{\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}}.$$

Dfn: If $\vec{u}, \vec{v} \in \mathbb{R}^n$, define the projection of $\vec{u}$ onto $\vec{v}$ by

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}.$$

$(\text{proj}_{\vec{v}} : \mathbb{R}^n \to \mathbb{R}^n)$



Claim: set $\vec{p} = \text{proj}_{\vec{v}}(\vec{u})$. Set $\vec{w} = \vec{u} - \vec{p}$

$\vec{w} \cdot \vec{v}$ should be $\vec{0}$.

$$\vec{w} \cdot \vec{v} = (\vec{u} - \vec{p}) \cdot \vec{v} = \vec{u} \cdot \vec{v} - \vec{p} \cdot \vec{v} = \vec{u} \cdot \vec{v} - \overbrace{\left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \right)}^{p} \cdot \vec{v}$$

$$= \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\cancel{\vec{v} \cdot \vec{v}}} \cancel{(\vec{v} \cdot \vec{v})} = 0.$$

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

$$\text{proj}_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\, \vec{v} = \frac{25}{50} \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix}.$$

$$\vec{w} = \vec{u} - \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix}$$

$$\vec{w} \cdot \vec{v} = \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 7 \end{bmatrix} = -\frac{7}{2} + \frac{7}{2} = 0.$$

§7.2 Inner Products

Want analogue of dot product

Dfn: V a VS. An <u>inner product</u> on V is an operation $\langle \vec{u}, \vec{v} \rangle : \vec{V} \times \vec{V} \to \mathbb{R}$ s.t.

$\langle \vec{u}, \vec{v} \rangle$

\langle

\rangle

1) (Positive Definite) $\langle \vec{u}, \vec{u} \rangle \geq 0$, and $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$

2) (Symmetry) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$.

3) (Bilinearity) $= \langle \alpha \vec{u} + \beta \vec{v}, \vec{w} \rangle = \alpha \langle \vec{u}, \vec{w} \rangle + \beta \langle \vec{v}, \vec{w} \rangle$.

we write $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ is the <u>norm</u> of $\vec{v}$.

$\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$

Ex: The dot product is an inner product

Ex: $V = \mathcal{C}([a,b], \mathbb{R})$. $\langle f, g \rangle = \int_a^b f(t) g(t)\, dt$

$\|f\|_2 = \sqrt{\int_a^b f(t)^2 dt}$

1) (PD) $\langle f, f \rangle = \int_a^b (f(t))^2 dt \geq 0$

if $0 = \langle f, f \rangle = \int_a^b (f(t))^2 dt$ so $f(t)^2 = 0$

so $f(t) = 0$

$L^2([a,b])$

2) (S) $\langle f, g \rangle = \int_a^b f(t)\, g(t) dt = \int_a^b g(t) f(t) dt = \langle g, f \rangle$

3) (Bil) $\langle \alpha f + \beta g, h \rangle = \int_a^b (\alpha f(t) + \beta g(t)) h(t)\, dt$

$= \alpha \int_a^b f(t)\, h(t) dt + \beta \int_a^b g(t)\, h(t) dt = \alpha \langle f, h \rangle + \beta \langle g, h \rangle.$

Ex: $V = P_n(x)$, fix $x_0, x_1, \ldots, x_n$ distinct real numbers

define $\langle f, g \rangle = \sum_{i=0}^{n} f(x_i) g(x_i) = f(x_0)g(x_0) + \cdots + f(x_n)\, g(x_n)$

1) (PD) $\langle f, f \rangle = f(x_0)^2 + f(x_1)^2 + \cdots + f(x_n)^2 \geq 0$

suppose $0 = \langle f, f \rangle = f(x_0)^2 + f(x_1)^2 + \cdots + f(x_n)^2$

$f(x_i)^2 = 0$ for each i, so $f(x_i) = 0$ for each i.

so f is a deg n poly with $n+1$ roots.

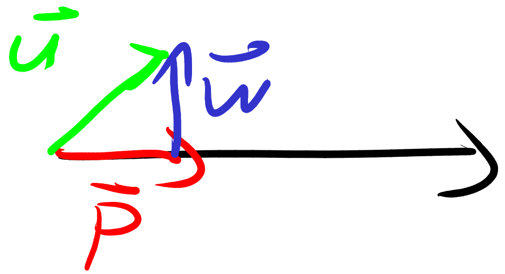
That can only happen if $f = 0$.

2) (S)
3) (Bil) HW

$\|f\| = \sqrt{f(x_0)^2 + f(x_1)^2 + \cdots + f(x_n)^2}$.

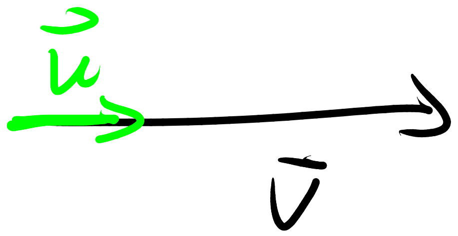
Dfn: $\vec{u}, \vec{v} \in V$. If $\langle \vec{u}, \vec{v} \rangle = 0$, say $\vec{u}, \vec{v}$ are <u>orthogonal</u>.

Dfn: $\vec{u}, \vec{v} \in V$, $\vec{v} \neq \vec{0}$. Define



$$\mathrm{proj}_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}.$$

Prop: $\vec{u}, \vec{v} \in V$, $\vec{v} \neq \vec{0}$. Set $\vec{p} = \mathrm{proj}_{\vec{v}} \vec{u}$. Then:



1) $\langle \vec{u} - \vec{p}, \vec{v} \rangle = \langle \vec{u} - \vec{p}, \vec{p} \rangle = 0$.

2) $\vec{u} = \vec{p}$ iff $\vec{u} = \beta \vec{v}$ for some $\beta \in \mathbb{R}$

Pf/ If $\vec{u} = \beta \vec{v}$, then $\vec{p} = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} = \beta \dfrac{\cancel{\langle \vec{v}, \vec{v} \rangle}}{\cancel{\langle \vec{v}, \vec{v} \rangle}} \vec{v} = \beta \vec{v} = \vec{u}$.

Now suppose $\vec{u} = \vec{p} = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$. Set $\beta = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}$.

Ex: $V = \mathcal{C}([-1,1], \mathbb{R})$, $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$.

$f(x) = 1$, $g(x) = x$

$\langle 1, x \rangle = \int_{-1}^{1} 1 \cdot t\,dt = 0$. $\quad$ 1, x are orthogonal.

$h(x) = 1 + x = f + g$

$\langle h, f \rangle = \int_{-1}^{1} 1 + t\,dt = 2$ $\quad$ $\text{proj}_f h = \frac{\langle h, f \rangle}{\langle f, f \rangle} f = \frac{2}{\langle f, f \rangle} \cdot 1 = 1$

$\langle f, f \rangle = \int_{-1}^{1} 1^2\,dt = 2$ $\quad$ $\|f\|_2 = \sqrt{2}$.

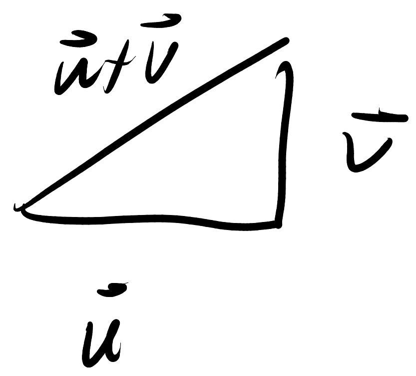
$\langle h, g \rangle = \int_{-1}^{1} t + t^2\,dt = \frac{2}{3}$ $\quad$ $\text{proj}_g h = \frac{\langle h, g \rangle}{\langle g, g \rangle} g = \frac{2/3}{2/3} \cdot x = x$.

$\langle g, g \rangle = \int_{-1}^{1} t^2\,dt = 2/3$

Prop (Pythagorean Law) if $\vec{u}, \vec{v}$ are orthogonal.

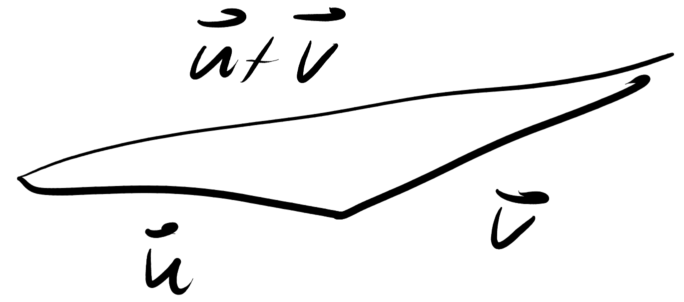
$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Pf/ H W. $\|\vec{u}\|^2 = \langle \vec{u}, \vec{u} \rangle$



Thm (Cauchy-Schwarz Inequality)

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|.$$



Pf/

Thm (Cauchy-Schwarz Inequality)

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|.$$

Pf/ Assume $\vec{v} \neq \vec{0}$. Set $\vec{p} = \text{proj}_{\vec{v}} \vec{u}$.

$$\|u\|^2 = \|\vec{p}\|^2 + \|\vec{u} - \vec{p}\|^2$$

$$\|p\|^2 = \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \right\|^2 = \frac{\langle \vec{u}, \vec{v} \rangle^2}{\langle \vec{v}, \vec{v} \rangle^2} \cancel{\langle \vec{v}, \vec{v} \rangle} = \frac{\langle \vec{u}, \vec{v} \rangle^2}{\langle \vec{v}, \vec{v} \rangle}$$

$$\frac{\langle \vec{u}, \vec{v} \rangle^2}{\langle \vec{v}, \vec{v} \rangle} = \|\vec{u}\|^2 - \|\vec{u} - \vec{p}\|^2$$

$$\langle \vec{u}, \vec{v} \rangle^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u} - \vec{p}\|^2 \|\vec{v}\|^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$$

$$\sqrt{\langle \vec{u}, \vec{v} \rangle^2} \leq \|\vec{u}\| \, \|\vec{v}\|.$$

Dfn: $\vec{u}, \vec{v} \in V$. Set the angle btwn them to be

$$\theta = \arccos\left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|}\right)$$