Inner Products

$\langle \vec{x}, \vec{y} \rangle : V \times V \to \mathbb{R}$

- $\langle \vec{x}, \vec{x} \rangle \geq 0$
- $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$
- $\langle \alpha \vec{x} + \beta \vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$

If $\langle \vec{x}, \vec{y} \rangle = 0$, $\vec{x} \perp \vec{y}$.

$$\mathrm{proj}_{\vec{y}} \vec{x} = \frac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle} \vec{y}$$

$$(\vec{x} - \mathrm{proj}_{\vec{y}} \vec{x}) \perp \vec{y}$$

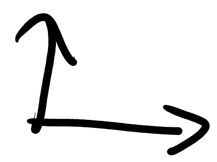
§7.3 Orthonormal basis

Dfn: $S = \{ \vec{u}_1, \dots, \vec{u}_n \} \subseteq V$

is <u>orthogonal</u> if $\vec{u}_i \perp \vec{u}_j$ for $i \neq j$

S is <u>orthonormal</u> if further $\|\vec{u}_i\| = 1$.

Prop: Any orthogonal set of non-zero vectors is LI.



$$\langle u_1+u_2, u_1+u_2\rangle = \langle u_1,u_1\rangle \cancel{+\langle u_1,u_2\rangle} \cancel{+\langle u_2,u_1\rangle} + \langle u_2,u_2\rangle$$

Pf/ Suppose $a_1\vec{u}_1 + \cdots + a_n\vec{u}_n = \bar{0}$

$$0 = \langle \vec{0}, \vec{0}\rangle = \langle a_1\vec{u}_1 + \cdots + a_n\vec{u}_n,\ a_1\vec{u}_1 + \cdots + a_n\vec{u}_n \rangle$$

$$= \sum_{i,j=1}^{n} a_i a_j \langle \vec{u}_i, \vec{u}_j\rangle = \sum_{i=1}^{n} a_i^2 \langle \vec{u}_i, \vec{u}_i\rangle$$

$$= \sum_{i=1}^{n} a_i^2 \|u_i\|^2.$$

so $a_i^2\|u_i\|^2 = 0 \ \forall i$

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so $a_i = 0 \ \forall i$.

$$\langle u_1+u_2, u_1+u_2\rangle = \langle u_1, u_1+u_2\rangle + \langle u_2, u_1+u_2\rangle$$

$$= \left(\langle u_1, u_1\rangle + \langle u_1, u_2\rangle\right) + \left(\langle u_2, u_1\rangle + \langle u_2, u_2\rangle\right)$$

$$= \langle u_1, u_1\rangle \cancel{+\langle u_1, u_2\rangle}$$

$$\cancel{+\langle u_2, u_1\rangle} + \langle u_2, u_2\rangle$$

Dfn: $E = \{\vec{e}_1, \ldots, \vec{e}_n\}$ basis for V.

E is an orthogonal basis if $\langle \vec{e}_i, \vec{e}_j \rangle = 0$ for $i \neq j$

E orthonormal if $\|\vec{e}_i\| = 1$.

Ex: Std basis for $\mathbb{R}^n$

Ex: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ orthog, not ON

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ON

$V = P_2(x)$, $\langle f, g\rangle = \int_{-1}^{1} f(t)\,g(t)\,dt$

$E = \{1, x, 3x^2 - 1\}$ Orthog basis

$\langle x, x\rangle = \int_{-1}^{1} x^2\,dx$

$= \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$

$\|x\| = \sqrt{2/3}$

$F = \left\{\frac{1}{\sqrt{2}}, \frac{x\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)\right\}$ ON

$V = P_2(x)$, $\langle f, g\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$

$E = \{1, x, x^2 - 2/3\}$ is orthog

$F = \left\{\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}(x^2 - 2/3)\right\}$.

$\|1\| = \sqrt{\langle 1, 1\rangle} = \sqrt{1+1+1} = \sqrt{3}$

$\langle 1, x^2\rangle = 2$

Prop: V VS, $E = \{\vec{e}_1, \dots, \vec{e}_n\}$ ON Basis

$$\vec{u} = \sum_{i=1}^{n} a_i \vec{e}_i, \quad \vec{v} = \sum_{i=1}^{n} b_i \vec{e}_i$$

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i,j=1}^{n} \langle a_i \vec{e}_i, b_j \vec{e}_j \rangle = \sum_{i=1}^{n} a_i b_i \langle \vec{e}_i, \vec{e}_i \rangle$$

$$= \sum_{i=1}^{n} a_i b_i$$ This is the dot product!

and $\|\vec{u}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

Prop: V a VS, $E = \{\vec{e}_1, \dots, \vec{e}_n\}$ orthog basis

If $\vec{v} \in V$, then

$$\vec{v} = \mathrm{proj}_{\vec{e}_1} \vec{v} + \mathrm{proj}_{\vec{e}_2} \vec{v} + \dots + \mathrm{proj}_{\vec{e}_n} \vec{v}.$$

Pf/ if $\vec{v} = a_1 \vec{e}_1 + \dots a_n \vec{e}_n$, then

$$\mathrm{proj}_{\vec{e}_1} \vec{v} = \frac{\langle \vec{v}, \vec{e}_1 \rangle}{\langle \vec{e}_1, \vec{e}_1 \rangle} \vec{e}_1$$

$$= \frac{a_1 \langle \vec{e}_1, \vec{e}_1 \rangle + \cancel{\dots + a_n \langle \vec{e}_n, \vec{e}_1 \rangle}}{\langle \vec{e}_1, \vec{e}_1 \rangle} \vec{e}_1 = \frac{a_1 \cancel{\langle \vec{e}_1, \vec{e}_1 \rangle}}{\cancel{\langle \vec{e}_1, \vec{e}_1 \rangle}} \vec{e}_1$$

Cor: V vs, $E = \{\vec{e}_1, \dots, \vec{e}_n\}$ ON basis, $\vec{v} \in V$.

Then $\vec{v} = \dfrac{\langle v, e_1 \rangle}{\cancel{\langle e_1, e_1 \rangle}} e_1 + \dots + \dfrac{\langle v, e_n \rangle}{\cancel{\langle e_n, e_n \rangle}} \vec{e_n}$

$$= \langle \vec{v}, \vec{e_1} \rangle \vec{e_1} + \dots + \langle \vec{v}, \vec{e_n} \rangle \vec{e_n}.$$

Fourier series $\quad C([-\pi, \pi], \mathbb{R})$, $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t)\, dt$

$\left\{ \frac{1}{2\pi}, \frac{\sin(nx)}{\pi}, \frac{\cos(nx)}{\pi} \right\}$ ON basis

$$F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \text{ find } \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}_F. \quad = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{proj}_{\vec{f}_1} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{8}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{proj}_{\vec{f}_2} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\text{proj}_{\vec{f}_3} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

E = std basis. find coords of $\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$$\text{proj}_{\vec{e}_1} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{6}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{proj}_{\vec{e}_2} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Goread 7.37.

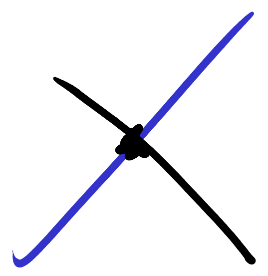
$$3x^2 - 6x + 4 = 5(1) - 6(x) + 1(3x^2 - 1)$$

Gram-Schmidt Process

turns basis $\longrightarrow$ OG basis $\longrightarrow$ ON basis

$$\left\{\begin{matrix} \vec{e}_1 \\ e_2 \\ \vdots \\ e_n \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \vec{e}_1 \\ \vec{e}_2 - \mathrm{proj}_{e_1} \vec{e}_2 \\ \vec{e}_3 - \mathrm{proj}_{\vec{e}_1} \vec{e}_3 - \mathrm{proj}_{f_2} \vec{e}_3 \\ \vdots \\ \vec{e}_n - \mathrm{proj}_{\vec{e}_1} \vec{e}_n - \mathrm{proj}_{f_2} \vec{e}_n \\ - \cdots - \mathrm{proj}_{\vec{f}_{n-1}} \vec{e}_n \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \vec{f}_1 / \|\vec{f}_1\| \\ \vec{f}_2 / \|\vec{f}_2\| \\ \vdots \\ \vec{f}_n / \|\vec{f}_n\| \end{matrix}\right\}$$

§7.4 Orthogonal Subspaces



Dfn: V an IP space, U, W subspaces, say U, W are <u>orthogonal</u> or $U \perp W$ if $\langle \vec{u}, \vec{w} \rangle = 0 \ \forall \vec{u} \in U, \vec{w} \in W$.

If $U \leq V$, the <u>orthogonal complement</u> of U is $U^{\perp} = \{\vec{v} \in V \mid \langle \vec{v}, \vec{u} \rangle = 0 \ \forall \vec{u} \in U\}$.

Prop: If $U \leq V$, then $U^\perp$ is a subspace of V.

Pf/ 1) $\vec{0} \perp \vec{u}$ for any $\vec{u}$.

2) If $\langle \vec{v}, \vec{u} \rangle = 0$ then $\langle r\vec{v}, \vec{u} \rangle = r \langle \vec{v}, \vec{u} \rangle = r\,0 = 0$.

3) If $v \perp u$, $w \perp u$, then $\langle \vec{v} + \vec{w}, \vec{u} \rangle = \langle \vec{v}, \vec{u} \rangle + \langle \vec{w}, \vec{u} \rangle = 0 + 0$.

Prop: A a matrix. $\ker(A) = (\operatorname{row}(A))^\perp$

Pf/ $\ker = \{ \vec{v} \mid A\vec{v} = \vec{0} \} = \left\{ \vec{v} \,\middle|\, \begin{bmatrix} \vec{r}_1 \cdot \vec{v} \\ \vec{r}_2 \cdot \vec{v} \\ \vdots \\ \vec{r}_n \cdot \vec{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$.

$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Find $U^\perp$

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 3 & 1 \end{bmatrix}$$

so $U^\perp = \ker(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \right\}$.

Prop: $U \leq V$, $\vec{v} \in V$; then $\exists$ unique vectors $\vec{v}_U \in U$, $\vec{v}_{U^\perp} \in U^\perp$ s.t. $\vec{v} = \vec{v}_U + \vec{v}_{U^\perp}$.

Cor: $\dim U + \dim U^\perp = \dim V$. (Rank-nullity thm again!)

Ex: decompose $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ wrt plane $x - y + 2z = 0$.

basis for $U^\perp$ is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

$$\operatorname{proj}_{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \frac{8}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}_{U^\perp}$$

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}_U = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}_{U^\perp} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$$

Big dataset.
What's important?

DW-Nominate

1) Find eigenvectors
2) project each person onto these eigenvectors