

4 Linear Functions

Now that we understand vector spaces a bit more, we want to see how functions between vector spaces work. There are of course lots of functions that take in vectors and output other vectors; almost any multivariable function technically qualifies. But we actually want to care about functions that in some sense are compatible with the actual vector space structure.

4.1 Definition and examples

Definition 4.1. Let U and V be vector spaces, and let $L : U \rightarrow V$ be a function with domain U and codomain V . We say L is a *linear transformation* if:

1. Whenever $\mathbf{u}_1, \mathbf{u}_2 \in U$, then $L(\mathbf{u}_1 + \mathbf{u}_2) = L(\mathbf{u}_1) + L(\mathbf{u}_2)$.
2. Whenever $\mathbf{u} \in U$ and $r \in \mathbb{R}$, then $L(r\mathbf{u}) = rL(\mathbf{u})$.

Example 4.2. If A is a $m \times n$ matrix, then A gives us a linear transformation from \mathbb{R}^n into \mathbb{R}^m , given by $A(\mathbf{x}) = A\mathbf{x}$. That is, our input is a (column) vector in \mathbb{R}^n , and our output is the vector in \mathbb{R}^m we get by multiplying our column vector by our matrix.

Geometrically, a linear transformation can stretch, rotate, and reflect, but it cannot bend or shift.

Example 4.3. Consider the function from \mathbb{R}^2 to \mathbb{R}^2 given by a rotation of ninety degrees counterclockwise. We can see by drawing pictures that the sum of two rotated vectors is the rotation of the sum of the vectors, and that the rotation of a stretched vector is the same as the stretch of a rotated vector. So this is a linear transformation.

Example 4.4. A *translation* is a function defined by $f(\mathbf{x}) = \mathbf{x} + \mathbf{u}$ for some fixed vector \mathbf{u} . (Geometrically, it corresponds to sliding or translating your input in the direction and distance of the vector \mathbf{u}).

This is *not* a linear transformation. For instance, $f(r\mathbf{x}) = r\mathbf{x} + \mathbf{u} \neq r(\mathbf{x} + \mathbf{u}) = rf(\mathbf{x})$ unless $\mathbf{u} = \mathbf{0}$.

Example 4.5. The function $f(x) = x^2$ is not a linear transformation from \mathbb{R} to \mathbb{R} , since $f(2x) = (2x)^2 = 4x^2 \neq 2x^2 = 2f(x)$.

Example 4.6. Define a function $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ by $L(x, y, z) = (x + y, 2z - x)$. We check:

$$\begin{aligned} L((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= L(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2 + y_1 + y_2, 2z_1 + 2z_2 - x_1 - x_2) \\ &= (x_1 + y_1, 2z_1 - x_1) + (x_2 + y_2, 2z_2 - x_2) \\ &= L(x_1, y_1, z_1) + L(x_2, y_2, z_2). \\ L(r(x, y, z)) &= L(rx, ry, rz) = (rx + ry, 2rz - rx) = \\ &= r(x + y, 2z - x) = rL(x, y, z). \end{aligned}$$

Thus L is a linear transformation by definition.

Definition 4.7. Let $L : U \rightarrow V$ be a linear transformation. If $\mathbf{u} \in U$ is a vector, we say the element $L(\mathbf{u}) \in V$ is the *image* of \mathbf{u} .

If $S \subset U$ then we define the image of S to be the set $L(S) = \{L(\mathbf{u}) : \mathbf{u} \in S\}$ to be the set of images of elements of S . We say the image of the entire set U is the *image* of the function L .

The *kernel* of L is the set $\ker(L) = \{\mathbf{u} \in U : L(\mathbf{u}) = \mathbf{0}\}$ of elements of U whose image is the zero vector.

Another way of thinking about linear transformations is that they send lines to lines. In particular, the image of a subspace under a linear transformation is always a subspace—thus the image of a line will be either a point or a line.

Proposition 4.8. *Let $L : U \rightarrow V$ be a linear transformation, and let $S \subseteq U$ be a subspace of U . Then:*

1. $\ker(L)$ is a subspace of U .
2. The image $L(S)$ of S is a subspace of V .

Proof. 1. See homework 6.

2. We use the subspace theorem:

- (a) We wish to show that $\mathbf{0} \in L(S)$. We claim in particular that $L(\mathbf{0}) = \mathbf{0}$: that is, the image of the zero vector in U must be the zero vector in V . Recall that $0 \cdot \mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in V$, so we have

$$L(\mathbf{0}) = L(0 \cdot \mathbf{0}) = 0L(\mathbf{0}) = \mathbf{0}.$$

Thus since S is a subspace we have $\mathbf{0} \in S$ and thus $\mathbf{0} \in L(S)$.

- (b) Suppose $\mathbf{v} \in L(S)$ and $r \in \mathbb{R}$. Then there is some $\mathbf{u} \in S$ with $L(\mathbf{u}) = \mathbf{v}$, and since S is a subspace we know that $r\mathbf{u} \in S$. Thus

$$r\mathbf{v} = rL(\mathbf{u}) = L(r\mathbf{u}) \in L(S).$$

- (c) Suppose $\mathbf{v}_1, \mathbf{v}_2 \in L(S)$. Then there exist $\mathbf{u}_1, \mathbf{u}_2 \in S$ such that $L(\mathbf{u}_1) = \mathbf{v}_1$ and $L(\mathbf{u}_2) = \mathbf{v}_2$. Since S is a subspace we know that $\mathbf{u}_1 + \mathbf{u}_2 \in S$. Then

$$\mathbf{v}_1 + \mathbf{v}_2 = L(\mathbf{u}_1) + L(\mathbf{u}_2) = L(\mathbf{u}_1 + \mathbf{u}_2) \in L(S).$$

□

Corollary 4.9. *If $L : U \rightarrow V$ is a linear transformation, then the image of L is a subspace of V .*

Example 4.10. In our geometric example of a ninety degree counterclockwise rotation, the kernel is just the origin—nothing gets mapped to the origin except the origin. The image is the entire plane.

Example 4.11. If A is a matrix, then the linear transformation of A has a kernel precisely equal to the nullspace of A , since the nullspace is the set of \mathbf{x} such that $A\mathbf{x} = \mathbf{0}$.

In section 4.2 we will see that the image of A is the span of the columns of A .

Example 4.12. Let $\mathcal{D}([a, b], \mathbb{R})$ be the space of continuously differentiable functions from the closed interval $[a, b]$ to the real line. Define the derivative operator $D : \mathcal{D}([a, b], \mathbb{R}) \rightarrow \mathcal{C}([a, b], \mathbb{R})$ by $D(f) = f'$. First we claim that D is a linear operator: we have that $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$, and $D(rf) = (rf)' + rf' = rD(f)$.

The kernel of D is the space of constant functions, which is a one-dimensional subspace. The image of D is actually a little hard to see, but it's actually the set of all continuous functions on $[a, b]$.

In other contexts we might write $\frac{d}{dx}$ instead of D for this linear transformation.

Example 4.13. Let $\mathcal{C}([a, b], \mathbb{R})$ be the set of all continuous functions on the closed interval $[a, b]$. The (indefinite) integral isn't quite a linear transformation, since there's an ambiguity in choice of constant. (This is what we mean when we say something is "not well defined": if I tell you to give me the integral of x^2 , you can't give me a specific function back so my question is not precise enough).

But the function $I(f) = \int_a^x f(t) dt$ is a linear transformation, since $\int_a^x (f+g)(t) dt = \int_a^x f(t) dt + \int_a^x g(t) dt$ and $\int_a^x rf(t) dt = r \int_a^x f(t) dt$. In this case the choice of a as the basepoint resolves the earlier ambiguity.

The kernel of I is the trivial vector space containing only the zero function. The image is again a bit hard to see, but works out to be the space of differentiable functions with the property that $F(a) = 0$.

This last example shows an important principle: our derivative and integral linear transformations (almost) undo each other. This is a very important property and we will look at it on its own in 4.4.

4.2 Row space, column space and nullspace

We saw that every $m \times n$ matrix gives a linear transformation from \mathbb{R}^n to \mathbb{R}^m . These linear transformations are easy to work with, because we know a lot about matrices. In this section we'll see what we can learn about those transformations. (In section 4.3 we'll see how to apply this to any linear transformation.)

Definition 4.14. If $A = (a_{ij})$ is a $m \times n$ matrix, then each row can be viewed as a vector in \mathbb{R}^n ; we call these vectors the *row vectors* of A . We may notate them as $\mathbf{r}_i = (a_{i1}, a_{i2}, \dots, a_{in})$.

Similarly, we can view each column as vector in \mathbb{R}^m , and we call these the *column vectors* of A . We may notate them as $\mathbf{c}_j = (a_{1j}, a_{2j}, \dots, a_{mj})$

Thus each matrix gives us two sets of vectors. We can look at these vectors and see which vector spaces they span.

Definition 4.15. If A is a $m \times n$ matrix, we say that the span of the row vectors of A is the *row space* of A , which we will sometimes denote $\text{row}(A)$. It is a subspace of \mathbb{R}^n . The dimension of the row space is the *rank* of A , denoted $\text{rk}(A)$.

The span of the column vectors of A is the *column space* of A , sometimes denoted $\text{col}(A)$.

Recall that we defined the *nullspace* of A to be the set $N(A) = \ker(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ of solutions to the associated homogeneous system of linear equations. We define the *nullity* of A to be the dimension of $N(A)$.

We want to relate these ideas to the kernel and image we discussed last section. We know that every linear transformation is a matrix, and every matrix is a linear transformation.

It's pretty clear that $N(A)$ is the kernel of the associated linear transformation. By definition, $N(A)$ is the set of \mathbf{x} such that $A\mathbf{x} = \mathbf{0}$, and this is just the definition of the kernel of the linear transformation. Thus we sometimes call $N(A)$ the *kernel* of the matrix A as well.

The image is a bit trickier, but still has a clear answer.

Proposition 4.16. *Let A be a $m \times n$ matrix and \mathbf{b} a vector in \mathbb{R}^m . Then the image of the linear transformation associated to A is the column space of A . That is, $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is in $\text{col}(A)$.*

Proof. The equation $A\mathbf{x} = \mathbf{b}$ is the same as the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

which we can rewrite as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}.$$

Thus the equation has a solution precisely when \mathbf{b} is in the span of the \mathbf{c}_i , which is the column space of A by definition. \square

Corollary 4.17. *The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if $\text{col}(A) = \mathbb{R}^m$, that is, the column vectors span \mathbb{R}^m .*

The system has a unique solution if and only if the column vectors are linearly independent.

Proof. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if every $\mathbf{b} \in \mathbb{R}^m$ is in the column space, that is, if the column vectors span \mathbb{R}^m .

The column vectors are linearly independent if and only if every vector in their span can be represented uniquely as a linear combination of the column vectors. \square

This is only a partial answer, though, since we don't have a way to figure out what the column space actually looks like. To learn about that, we shift to looking at the row space, which is somewhat easier to understand.

Corollary 4.18. *Suppose A is a $m \times n$ matrix and A_R is the matrix obtained by using Gauss-Jordan elimination to reduce it to reduced row echelon form. Then the non-zero rows of A_R form a basis for the row space of A .*

Proof. The non-zero rows of A_R are clearly linearly independent, since each one has a 1 in a column where every other row has a zero. Thus the non-zero rows of A_R form a basis for the space they span, which is the row space of A_R . But we saw in section 1.3 that A_R and A have the same row space, so clearly the rows of A_R span the row space of A . Thus they form a basis for the row space of A . \square

Example 4.19. Find a basis for the row space of

$$\begin{bmatrix} 1 & 5 & -9 & 11 \\ -2 & -9 & 15 & -21 \\ 3 & 17 & -30 & 36 \\ -1 & 2 & -3 & -1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 5 & -9 & 11 \\ -2 & -9 & 15 & -21 \\ 3 & 17 & -30 & 36 \\ -1 & 2 & -3 & -1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 5 & -9 & 11 \\ 0 & 1 & -3 & 1 \\ 0 & 2 & -3 & 3 \\ 0 & 7 & -12 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 6 & 6 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 9 & 3 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 6 & 6 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 9 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So a basis for $\text{row}(A)$ is $\{(1, 0, 0, 4), (0, 1, 0, 2), (0, 0, 1, 1/3)\}$. The matrix has rank 3.

Remark 4.20. We can use this to find a “simple” basis for any vector space we have a spanning set for: write a matrix with our spanning set as rows, and row-reduce it until we have a basis.

As a consequence of this approach to the row space, we get an extremely powerful result relating the rank and the nullity:

Theorem 4.21 (Rank-Nullity). *If $A \in M_{m \times n}$ then rank of A plus nullity of A equals n .*

Proof. If U is the reduced row echelon form of A , then $A\mathbf{x} = \mathbf{0}$ is equivalent to $U\mathbf{x} = \mathbf{0}$. Since the matrix has rank r , the matrix U will have r nonzero rows and $n - r$ zero rows; thus it will have $n - r$ free variables and r lead variables.

The dimension of $N(A)$ is equal to the number of free variables, and thus to $n - r$. \square

We have managed to relate the rank and the nullity, but we still want to know about the column space. But the column space is tied to the row space in a fundamental way.

Proposition 4.22. *If A is a $m \times n$ matrix, the dimension of the row space of A equals the dimension of the column space of A .*

Proof. We will use a trick with the transpose matrix, since the rows of A are the columns of A^T and vice versa. We will prove that the dimension of the column space of a matrix is at least as great as the dimension of the row space. But since this result will also hold for the transpose matrix, this gives us our answer.

Suppose A has rank r , and let U be the row echelon form of A . It will have r leading 1s, and the columns containing the leading 1s will be linearly independent. (They do not form a basis for the column space, since we have no reason to believe that the row operations preserve the span of the *columns*).

Let U_L be the matrix obtained by deleting the columns of U corresponding to free variables, leaving only the columns that contain a leading 1. Delete the same columns from A , and call the resulting matrix A_L .

The matrices U_L and A_L are row-equivalent, so $A_L \mathbf{x} = \mathbf{0}$ if and only if $U_L \mathbf{x} = \mathbf{0}$, and since the columns of U_L are linearly independent, this happens if and only if $\mathbf{x} = \mathbf{0}$. Thus we see that the columns of A_L are linearly independent. We know that A_L will have exactly r columns, so the column space contains at least r linearly independent vectors, and so the dimension of the column space is at least r . Thus $\dim(\text{col}(A)) \geq \dim(\text{row}(A)) = r$.

Now consider the matrix A^T . By the previous result, $\dim(\text{col}(A^T)) \geq \dim(\text{row}(A^T))$. But we know that $\text{col}(A^T) = \text{row}(A)$ and $\text{row}(A^T) = \text{col}(A)$, so this tells us that $\dim(\text{row}(A)) \geq \dim(\text{col}(A))$, which combined with the previous result gives us that $\dim(\text{row}(A)) = \dim(\text{col}(A))$. □

Remark 4.23. This gives us another perspective on our attempts to figure out whether a set of vectors was a spanning set. If we write a matrix whose rows are our vectors, then the rank is the dimension of the rowspace; so the rows span if and only if the rank is equal to the dimension of the space, which is the length of each row, which is the number of columns.

Alternatively, we can make a matrix whose columns are the vectors. Then the rank is the dimension of the column space, so our vectors span if and only if the rank is equal to the height of the columns, which is the number of rows.

We still need a way to actually find the image, however. Fortunately, the proof of proposition 4.22 gives us a way.

Corollary 4.24. *Let A be a $m \times n$ matrix, and let U be the reduced row echelon form of A . Then the columns of A corresponding to columns of U that contain a leading “1” form a basis for the column space of A .*

Proof. We just showed that these columns are linearly independent, and there are r of them. Thus they are a basis. \square

Remark 4.25. Note that the columns of U do not (usually) span the column space of A ! But looking at U tells us which columns we should take to find a basis for the column space.

Note that we could also find a basis for the column space by simply taking A^T , row reducing it, and finding a basis for the row space of A^T .

Example 4.26. Find a basis for the column space of
$$\begin{bmatrix} 1 & 5 & -9 & 11 \\ -2 & -9 & 15 & -21 \\ 3 & 17 & -30 & 36 \\ -1 & 2 & -3 & -1 \end{bmatrix}$$

We saw that the reduced row echelon form of this matrix has leading ones in the first three columns. So the first three columns form a basis for the column space, and thus a basis is $\{(1, -2, 3, -1), (5, -9, 17, 2), (-9, 15, -30, -3)\}$.

Example 4.27. Find bases for the row, column, and nullspace of
$$\begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$$

We first row reduce the matrix.

$$\begin{aligned} \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 4 & 4 & 12 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

To find the row space, we just take these rows; so a basis for the row space is $\{(1, 0, 3, 7, 0), (0, 1, 1, 3, 0), (0, 0, 0, 0, 1)\}$. Thus the rank of the matrix is 3.

To find the column space, we look at the columns corresponding to those with leading 1s, which are the first, second, and fifth. Thus a basis for the column space is

$$\{(1, -1, 0, 1), (-2, 3, 1, 2), (2, -2, 4, 5)\}.$$

To find the nullspace, we see there are two free variables, which we set to be parameters $x_3 = \alpha, x_4 = \beta$. Then the nullspace is

$$\begin{aligned} \{(-3\alpha - 7\beta, -\alpha - 3\beta, \alpha, \beta, 0)\} &= \{(-3\alpha, -\alpha, \alpha, 0, 0) + (-7\beta, -3\beta, 0, \beta, 0)\} \\ &= \{\alpha(-3, -1, 1, 0, 0) + \beta(-7, -3, 0, 1, 0)\} \end{aligned}$$

so a basis for the nullspace is $\{(-3, -1, 1, 0, 0), (-7, -3, 0, 1, 0)\}$. The nullity is 2, which is what we expected from the rank-nullity theorem.

4.3 The Matrix of a Linear Transformation

We now know a lot about linear transformations that come from matrices. But what about other linear transformations? It turns out that this isn't really an issue, because it turns out that *all* linear transformations (of finite-dimensional vector spaces) come from matrices.

In essence, we can represent a linear transformation $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with a matrix because we have a system of coordinates for \mathbb{R}^n and \mathbb{R}^m ; the matrix tells us what happens to each coordinate.

Example 4.28. Let $A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix}$ be a matrix, and thus a linear transformation $\mathbb{R}^3 \rightarrow \mathbb{R}^2$. Let's see what happens to each element of the standard basis for \mathbb{R}^3 .

$$\begin{aligned} A\mathbf{e}_1 &= \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ A\mathbf{e}_2 &= \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \\ A\mathbf{e}_3 &= \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \end{aligned}$$

We notice that the image of the standard basis elements are just the columns of the matrix! This isn't a coincidence; the columns of our matrix are telling us exactly where our basis vectors go.

Proposition 4.29. *Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$.*

In particular, the j th column vector of A is given by $\mathbf{c}_j = L(\mathbf{e}_j)$.

Proof. According to the theorem statement, we know that $A = [\mathbf{c}_1 \ \mathbf{c}_2 \ \dots \ \mathbf{c}_n]$. So we just need to check that this matrix gives us the linear transformation L .

First we show that our matrix does the right things on the standard basis vectors. We see that

$$A\mathbf{e}_j = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_j & \dots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{c}_j = L(\mathbf{e}_j).$$

Now let $\mathbf{u} \in \mathbb{R}^n$ be any vector. Then we know we can write $\mathbf{u} = \sum_{i=1}^n u_i \mathbf{e}_i$ since every element is some linear combination of basis vectors. Thus we have

$$\begin{aligned} A\mathbf{u} &= A \left(\sum_{i=1}^n u_i \mathbf{e}_i \right) = \sum_{i=1}^n Au_i \mathbf{e}_i = \sum_{i=1}^n u_i A\mathbf{e}_i = \sum_{i=1}^n u_i L(\mathbf{e}_i) \quad \text{by the previous computation} \\ &= \sum_{i=1}^n L(u_i \mathbf{e}_i) \quad \text{scalar multiplication} \\ &= L \left(\sum_{i=1}^n u_i \mathbf{e}_i \right) \quad \text{additivity} \\ &= L(\mathbf{u}). \end{aligned}$$

□

Example 4.30. Let's look at the linear transformation from earlier, of a 90 degree rotation counterclockwise. This is a transformation from \mathbb{R}^2 to \mathbb{R}^2 , so we can find a 2×2 matrix representing it. Let's call the map $R_{\pi/2}$.

By geometry, we see that $R_{\pi/2}(\mathbf{e}_1) = (0, 1) = \mathbf{e}_2$, and that $R_{\pi/2}(\mathbf{e}_2) = (-1, 0) = -\mathbf{e}_1$. Thus the matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Let's generalize to any rotation; let R_θ be the rotation counterclockwise by θ . To see what happens we have to draw the unit circle; we compute that $R_\theta(\mathbf{e}_1) = (\cos \theta, \sin \theta)$, and $R_\theta(\mathbf{e}_2) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2)) = (-\sin(\theta), \cos(\theta))$. Thus the matrix of R_θ is $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

Example 4.31. Define a linear transformation $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $L(x, y) = (x + y, x - y, 2x)$. First we should check that this is in fact a linear transformation, but I won't do that here.

We need to check the image of \mathbf{e}_1 and \mathbf{e}_2 . We see that

$$\begin{aligned} L(\mathbf{e}_1) &= L(1, 0) = (1, 1, 2) \\ L(\mathbf{e}_2) &= L(0, 1) = (1, -1, 0). \end{aligned}$$

Thus the matrix of L is

$$A_L = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

We can check this by computing

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \\ 2x \end{bmatrix}$$

which is exactly what we should get.

We'd like to be able to do this to any vector space, or at least any finite dimensional one. We need some set of coordinates to let us matricize other linear transformations. Fortunately, we developed those in section 3: a set of coordinates is a basis.

Definition 4.32. If U is a vector space and $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for U , and $\mathbf{u} \in U$, we can write $\mathbf{u} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$. We define the *coordinate vector* of \mathbf{u} with respect to E by

$$[\mathbf{u}]_E = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The a_i are called the *coordinates* of \mathbf{u} with respect to the basis E .

We here observe that every $\mathbf{u} \in U$ corresponds to exactly one coordinate vector with respect to E , and vice versa. We will discuss this in more detail in 4.4.

Example 4.33. Let $U = \mathcal{P}_3(x)$. Then $E = \{1, x, x^2, x^3\}$ is a basis for U . Also, $F = \{1, 1 + x, 1 + x^2, 1 + x^3\}$ is a basis for U .

Let $f(x) = 1 + 3x + x^2 - x^3 \in U$. Then

$$[f]_E = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix} \quad [f]_F = \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

These are two different vectors of real numbers, but they represent the *same* element of U , just in different bases.

Example 4.34. Let $U = \mathbb{R}^3$ and let $E = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$. Then if $\mathbf{u} = (1, 3, 2)$, then

$$[\mathbf{u}]_E = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Remark 4.35. If B is the standard basis for \mathbb{R}^n , then any time we write a column vector there's an implicit $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_B$ that we just don't bother to write down.

Lemma 4.36. *If U is a vector space and $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for U , then the function $[\cdot]_E : U \rightarrow \mathbb{R}^n$ which sends \mathbf{u} to $[\mathbf{u}]_E$ is a linear function.*

Proof. Let $\mathbf{u}, \mathbf{v} \in U$ and $r \in \mathbb{R}$. We can write

$$\mathbf{u} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$$

$$\mathbf{v} = b_1\mathbf{e}_1 + \dots + b_n\mathbf{e}_n.$$

Then

$$[r\mathbf{u}] = [ra_1\mathbf{e}_1 + \dots + ra_n\mathbf{e}_n] = (ra_1, \dots, ra_n) = r(a_1, \dots, a_n) = r[\mathbf{u}].$$

$$\begin{aligned} [\mathbf{u} + \mathbf{v}] &= [(a_1 + b_1)\mathbf{e}_1 + \dots + (a_n + b_n)\mathbf{e}_n] = (a_1 + b_1, \dots, a_n + b_n) \\ &= (a_1, \dots, a_n) + (b_1, \dots, b_n) = [\mathbf{u}] + [\mathbf{v}]. \end{aligned}$$

Thus by definition, $[\cdot]_E$ is a linear transformation. □

Theorem 4.37. *Let U and V be finite-dimensional vector spaces, with $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ a basis for U and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ a basis for V . Let $L : U \rightarrow V$ be a linear transformation.*

Then there is a matrix A that represents L with respect to E and F , such that $L\mathbf{u} = \mathbf{v}$ if and only if $A[\mathbf{u}]_E = [\mathbf{v}]_F$. The columns of A are given by $\mathbf{c}_j = [L(\mathbf{e}_j)]_F$.

Remark 4.38. This looks really complicated, but it really just says that any linear transformation is determined entirely by what it does to the elements of some basis; if you have a basis and you know where your transformation sends each element of that basis, you know what it does to everything in your space.

In particular, if we have coordinates for our vector spaces, we can use a matrix to map one set of coordinates to the other, as if we were working in \mathbb{R}^n .

$$\begin{array}{ccc} U & \xrightarrow{L} & V \\ \downarrow [\cdot]_E & & \downarrow [\cdot]_F \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \end{array} \quad \begin{array}{ccc} \mathbf{u} & \xrightarrow{L} & L(\mathbf{u}) \\ \downarrow [\cdot]_E & & \downarrow [\cdot]_F \\ [\mathbf{u}]_E & \xrightarrow{A} & A[\mathbf{u}]_E = [L(\mathbf{u})]_F \end{array}$$

Proof. We just want to show that $A[\mathbf{u}]_E = [L(\mathbf{u})]_F$ for any $\mathbf{u} \in U$, where

$$A = [\mathbf{c}_1 \dots \mathbf{c}_n] = [[L(\mathbf{e}_1)]_F \dots [L(\mathbf{e}_n)]_F].$$

Our proof is essentially the same as the proof of Proposition 4.29. Let $\mathbf{u} \in U$. Since E is a basis for U we can write $u = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$. Then we have

$$\begin{aligned} [L(\mathbf{u})]_F &= [a_1L(\mathbf{e}_1) + \dots + a_nL(\mathbf{e}_n)]_F = a_1 [L(\mathbf{e}_1)]_F + \dots + a_n [L(\mathbf{e}_n)]_F \\ &= a_1\mathbf{c}_1 + \dots + a_n\mathbf{c}_n; \\ A[\mathbf{u}]_E &= A[a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n]_E = A(a_1, \dots, a_n) = [\mathbf{c}_1 \dots \mathbf{c}_n](a_1, \dots, a_n) \\ &= \mathbf{c}_1a_1 + \dots + \mathbf{c}_na_n. \end{aligned}$$

Thus we have $[L(\mathbf{u})]_F = A[\mathbf{u}]_E$, so the matrix A does in fact represent the linear operator L . \square

Example 4.39. Let $F = \{(1, 1), (-1, 1)\}$ be a basis for \mathbb{R}^2 , and let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by $L(x, y, z) = (x - y - z, x + y + z)$. Find a matrix for L with respect to the standard basis in the domain and F in the codomain.

$$L(1, 0, 0) = (1, 1) = \mathbf{f}_1$$

$$L(0, 1, 0) = (-1, 1) = \mathbf{f}_2$$

$$L(0, 0, 1) = (-1, 1) = \mathbf{f}_2$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Example 4.40. Let S be the subspace of $\mathcal{C}([a, b], \mathbb{R})$ spanned by $\{e^x, xe^x, x^2e^x\}$, and let D be the differentiation operator on S . Find the matrix of D with respect to $\{e^x, xe^x, x^2e^x\}$.

We compute:

$$\begin{aligned} D(e^x) &= e^x = \mathbf{s}_1 \\ D(xe^x) &= e^x + xe^x = \mathbf{s}_1 + \mathbf{s}_2 \\ D(x^2e^x) &= 2xe^x + x^2e^x = 2\mathbf{s}_2 + \mathbf{s}_3 \\ A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Example 4.41. Let $E = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ be bases for \mathbb{R}^3 , and define $L(x, y, z) = (x + y + z, 2z, -x + y + z)$. We can check this is a linear transformation.

To find the matrix of L with respect to E and the standard basis, we compute

$$\begin{aligned} L(1, 1, 0) &= (2, 0, 0) \\ L(1, 0, 1) &= (2, 2, 0) \\ L(0, 1, 1) &= (2, 2, 2). \end{aligned}$$

Thus the matrix with respect to E and the standard basis is

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

If we want to find the matrix with respect to E and F , we observe that

$$\begin{aligned} L(1, 1, 0) &= (2, 0, 0) = 2(1, 0, 0) = 2\mathbf{f}_1 \\ L(1, 0, 1) &= (2, 2, 0) = 2(1, 1, 0) = 2\mathbf{f}_2 \\ L(0, 1, 1) &= (2, 2, 2) = 2(1, 1, 1) = 2\mathbf{f}_3. \end{aligned}$$

Thus the matrix is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We notice that this matrix is really simple; this is a “good” choice of bases for this linear transformation.

In contrast, let's look at the transformation $T(x, y, z) = (x, y, z)$. Then we have

$$T(1, 1, 0) = (1, 1, 0) = (1, 1, 0) = \mathbf{f}_2$$

$$T(1, 0, 1) = (1, 0, 1) = (1, 0, 0) - (1, 1, 0) + (1, 1, 1) = \mathbf{f}_1 - \mathbf{f}_2 + \mathbf{f}_3$$

$$T(0, 1, 1) = (0, 1, 1) = -(1, 0, 0) + (1, 1, 1) = -\mathbf{f}_1 + \mathbf{f}_3.$$

Thus the matrix of T with respect to E and F is

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus this transformation, which is really simple with respect to the standard basis, is much more complicated with respect to these bases.

We'll talk a lot more about this choice of basis idea in section 6, and we'll talk about how to find the best basis for a given linear transformation in section 5.

This allows us to translate most of the results from section 4.2 to tell us about any linear transformation of vector spaces. The image of a transformation corresponds to the column space of the associated matrix; the kernel corresponds to the nullspace of the associated matrix.

Then this gives us a generalization of the rank-nullity theorem: the rank is the dimension of the row space, which is the dimension of the column space, which is the dimension of the image. And the nullity is the dimension of the nullspace, which is the dimension of the kernel. The rank-nullity theorem 4.21 tells us the rank and nullity add up to the number of columns of the associated matrix—which is the dimension of the domain of the linear transformation. All combined, this gives us

Theorem 4.42 (Rank-Nullity for Vector Spaces). *Let U, V be finite-dimensional vector spaces, and $L : U \rightarrow V$ be a linear transformation. Then $\dim \ker(L) + \dim \operatorname{Im}(L) = \dim U$.*

Example 4.43. Define $L : \mathcal{P}_3(x) \rightarrow \mathcal{P}_3(x)$ be given by $L(f) = (1+x)f'' - f'$. We can take a standard basis $\{(1, x, x^2, x^3)\}$ and compute the matrix:

$$\begin{aligned} L(1) &= 0 & L(x) &= -1 \\ L(x^2) &= (1+x)2 - 2x = 2 & L(x^3) &= (1+x)6x - 3x^2 = 6x + 3x^2 \end{aligned}$$

so we get the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is already pretty close to row-reduced, but we can finish it off and get the reduction

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has rank 2 and nullity 2. Thus the dimension of the image is 2; it's spanned by the second and fourth columns of the original matrix, which correspond to $\{-1, 6x + 3x^2\}$. And the dimension of the kernel is 2. The kernel satisfies $a_1 + a_2 = 0, a_3 = 0$, so it's the set $\{a_0 + a_1x - a_1x^2\}$.

As a final result, we will see that we now know about every possible subspace. We know that the kernel of a linear transformation is a subspace; but the converse is true as well, and every subspace is the kernel of some linear transformation.

Proposition 4.44. *Let V be a vector space and $U \subset V$ a subspace. Then U is the kernel of some linear transformation.*

Proof. We'll prove this in the case where U and V are finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for U . By basis padding, there is a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}_1, \dots, \mathbf{v}_m\}$ for the vector space V .

Define a linear transformation $L : V \rightarrow V$ by setting $L(\mathbf{u}_i) = \mathbf{0}$ and $L(\mathbf{v}_i) = \mathbf{v}_i$. That is, for any $\mathbf{v} \in V$, we can write

$$v = a_1\mathbf{u}_1 + \dots + a_n\mathbf{u}_n + b_1\mathbf{v}_1 + \dots + b_m\mathbf{v}_m,$$

so we define

$$L(\mathbf{v}) = b_1\mathbf{v}_1 + \dots + b_m\mathbf{v}_m.$$

Then the kernel of L is the set spanned by $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$, which is just U . □

4.4 Isomorphisms

In the previous section, we looked at a lot of linear transformations. In particular, we saw the coordinate map $[\cdot]_E : U \rightarrow \mathbb{R}^n$ which sends a vector $\mathbf{u} \in U$ to its coordinates with respect to E . We observed that this mapping in fact goes both ways: if we have a vector we can compute the coordinates, and if we have coordinates we can compute the vectors. Functions like this are very important and have a special name.

Definition 4.45. Let $f : U \rightarrow V$ be a function. If there is a $g : V \rightarrow U$ such that $g(f(\mathbf{u})) = \mathbf{u}$ for all $u \in U$, and $f(g(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v} \in V$, then we say that $g = f^{-1}$ is the *inverse* of f , and that f is *invertible*.

If f is an invertible linear transformation, we say that f is an *isomorphism* between U and V .

If U and V are vector spaces, we say they are *isomorphic* if there exists an isomorphism from U to V . We write $U \cong V$.

Remark 4.46. We will see that if two spaces are isomorphic, we can treat them as essentially the same. This does not mean they are the same; \mathbb{R}^5 is not the same thing as the space of degree-four polynomials.

But if $U \cong V$ then they are the same *as vector spaces*, because if we want to do something to U , we can instead map it to V , do it there, and then map it back.

Example 4.47. Let U be a vector space with basis $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, and let $f : U \rightarrow \mathbb{R}^n$ be defined by $f(\mathbf{u}) = [\mathbf{u}]_E$. Then f is invertible, and the inverse of f is given by the function $g(a_1, \dots, a_n) = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$. To prove this, we check two things.

For any $\mathbf{u} \in U$ we can write $\mathbf{u} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$. Then

$$g(f(\mathbf{u})) = g(f(a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n)) = g(a_1, \dots, a_n) = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n = \mathbf{u}.$$

Similarly, for any $(a_1, \dots, a_n) \in \mathbb{R}^n$ we have

$$f(g(a_1, \dots, a_n)) = f(a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n) = (a_1, \dots, a_n).$$

Thus g is the inverse of f by definition.

Example 4.48. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $f(x, y) = (x + y, x - y)$. Then f is invertible, with inverse $g(a, b) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$. To prove this we check:

$$g(f(x, y)) = g(x + y, x - y) = \left(\frac{(x + y) + (x - y)}{2}, \frac{(x + y) - (x - y)}{2}\right) = (x, y)$$

$$f(g(a, b)) = f\left(\frac{a + b}{2}, \frac{a - b}{2}\right) = \left(\frac{a + b}{2} + \frac{a - b}{2}, \frac{a + b}{2} - \frac{a - b}{2}\right) = (a, b).$$

Thus g is the inverse of f by definition.

So far we can check whether a given g is the inverse of f , but we don't have a good way of determining if a function is invertible. In order to do that, we have to recall a few properties of functions. (Discrete Math covers these topics in much more detail; we need much less knowledge and information about them).

Definition 4.49. • A function f is *one-to-one* or *injective* if it has the property that: if $f(x) = f(y)$ then $x = y$. This tells us that anything in the image of f is only in the image once.

- A function $f : A \rightarrow B$ is *onto* or *surjective* if the image of f is B . That is, f is onto if for every $b \in B$ there is an $a \in A$ with $f(a) = b$. This tells us we can reach every element of the codomain from some element of the domain.
- A function f is *bijective* if it is both one-to-one and onto.

Proposition 4.50. Let $L : U \rightarrow V$ be a linear transformation of vector spaces. Then:

1. L is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.
2. L is invertible if and only if L is bijective.

Proof. 1. See Homework 8.

2. Suppose L is bijective. Define a transformation $T : V \rightarrow U$ as follows: let $\mathbf{v} \in V$. Then L is onto, so by definition there is some $\mathbf{u} \in U$ such that $L(\mathbf{u}) = \mathbf{v}$. Since L is one-to-one there is only one such element, since if $L(\mathbf{u}) = L(\mathbf{u}_1)$ then $\mathbf{u} = \mathbf{u}_1$ by definition of one-to-one. Define $T(\mathbf{v}) = \mathbf{u}$.

Then for any $\mathbf{v} \in V$, we have $L(T(\mathbf{v})) = L(\mathbf{u}) = \mathbf{v}$, and for any $\mathbf{u} \in U$ we have $T(L(\mathbf{u})) = T(\mathbf{v}) = \mathbf{u}$. Thus by definition, $T = L^{-1}$.

Conversely, suppose L is invertible, and let $T = L^{-1}$. Suppose $L(\mathbf{u}) = L(\mathbf{v})$. Then $T(L(\mathbf{u})) = T(L(\mathbf{v}))$, but $T(L(\mathbf{u})) = \mathbf{u}$ and $T(L(\mathbf{v})) = \mathbf{v}$, so $\mathbf{u} = \mathbf{v}$, and by definition L is one-to-one.

Let $\mathbf{v} \in V$. Then $T(\mathbf{v}) \in U$, and $L(T(\mathbf{v})) = \mathbf{v}$, so $\mathbf{v} \in L(U)$ for any $\mathbf{v} \in V$. Thus L is onto by definition, and since it is one-to-one and onto, it is bijective.

□

Remark 4.51. Another way to think of the second result is that “onto” guarantees that every element $\mathbf{v} \in V$ has an inverse, and “one-to-one” guarantees that no element has more than one, so the inverse function is actually well-defined as a function.

We can contrast with, say, the function $f(x) = x^2$. This function is not one-to-one, so when we ask for the inverse or square root of 4, we get two possible answers. (This function isn’t linear, but we didn’t actually use linearity anywhere in the previous proof, and in fact it works for all functions).

Proposition 4.50 gives us an easy way to check if a linear function is injective, and if we also check that it is surjective we can easily see whether it is invertible. We can always check surjectivity directly, and on your homework you will, but we’d like to make this easier as well. To do that we want to convert the Rank-Nullity Theorem to discuss all linear transformations. First we need to lay some groundwork.

Recall the rank-nullity theorem:

Theorem 4.52 (Rank-Nullity Theorem). *If U, V are finite-dimensional, then $\dim U = \dim \ker(L) + \dim L(U)$.*

Corollary 4.53. *Let $L : U \rightarrow V$ be linear. Then L is one-to-one if and only if $\dim U = \dim L(U)$, and L is onto if and only if $\dim V = \dim(U) - \dim \ker(L)$.*

If $\dim U = \dim V$, L is an isomorphism if and only if $\ker(L) = \{\mathbf{0}\}$.

Now we can easily determine if a transformation is invertible. But how do we find the inverse? Like everything in linear algebra, it’s easier to do computations if we change things to be a matrix.

Proposition 4.54. *Let $f : U \rightarrow V$ be a linear transformation of finite dimensional vector spaces, and let E, F be bases for U, V respectively. Let A be the matrix of f with respect to E, F . Then f is invertible if and only if A is invertible, and the matrix of f^{-1} is A^{-1} .*

Proof. Suppose f is invertible, and that the matrix of f is A and the Let B be the matrix of f^{-1} . Then for any $\mathbf{u} \in U$,

$$[f^{-1}(f(\mathbf{u}))]_E = B[f(\mathbf{u})]_F = BA[\mathbf{u}]_E$$

$$[f^{-1}(f(\mathbf{u}))]_E = [\mathbf{u}]_E$$

and thus $BA[\mathbf{u}]_E = [\mathbf{u}]_E$ for all $\mathbf{u} \in U$. Thus $B\mathbf{A}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, and thus $BA = I_n$. So by definition $B = A^{-1}$.

Conversely, suppose the matrix of f is A , and A has an inverse A^{-1} . Let g be the function corresponding to A^{-1} , so for all $\mathbf{v} \in V$ we have $[g(\mathbf{v})]_E = A^{-1}[\mathbf{v}]_F$. Then for any $\mathbf{u} \in U, \mathbf{v} \in V$, we compute

$$\begin{aligned} [g(f(\mathbf{u}))]_E &= A^{-1}[f(\mathbf{u})]_F = A^{-1}A[\mathbf{u}]_E = [\mathbf{u}]_E \\ [f(g(\mathbf{v}))]_F &= A[g(\mathbf{v})]_E = AA^{-1}[\mathbf{v}]_F = [\mathbf{v}]_F. \end{aligned}$$

Thus $g(f(\mathbf{u})) = \mathbf{u}$ and $f(g(\mathbf{v})) = \mathbf{v}$, so by definition $g = f^{-1}$. □

Corollary 4.55. *A $n \times n$ matrix is invertible if and only if its nullspace is trivial.*

This gives us a method for finding the inverse of any linear transformation: we find the matrix of the transformation, use Gaussian elimination to invert the matrix, and then return to the corresponding transformation.

Example 4.56. Let $L(x, y, z) = (x + y + z, 2x + 3y + 2z, x + 5y + 4z)$. What is L^{-1} ?

The matrix for L (with respect to the standard basis) is $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 5 & 4 \end{bmatrix}$. So we compute

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 3 & -1 & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 3 & 7 & -4 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/3 & -4/3 & 1/3 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/3 & -4/3 & 1/3 \end{array} \right] \end{aligned}$$

Since A is invertible, this tells us that L is also invertible. And from A^{-1} we can see that

$$L^{-1}(a, b, c) = (2a/3 + b/3 - c/3, -2a + b, 7a/3 - 4b/3 + c/3).$$

We can check by multiplying the original matrices together and seeing that we get the identity, or by computing $L^{-1}(L(x, y, z))$ and confirming that we get (x, y, z) .

Example 4.57. Let $E : \mathcal{P}_2(x) \rightarrow \mathbb{R}^3$ be given by $E(f(x)) = (f(-1), f(0), f(1))$. Can we find an inverse for this function?

Let $\{1, x, x^2\}$ be the basis for $\mathcal{P}_2(x)$, and use the standard basis for \mathbb{R}^3 . Then the matrix of this transformation is $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. To find the inverse, we compute

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array} \right]. \end{aligned}$$

We can double-check our work, again, by multiplying out the original matrices.

What have we concluded? If we have a quadratic polynomial such that $f(-1) = a$, $f(0) = b$, $f(1) = 3$, then we must have

$$f(x) = b + (c/2 - a/2)x + (a/2 + c/2 - b)x^2.$$

Thus we can use this technique to find the (minimal degree) polynomial that goes through a given set of points.

Example 4.58. Let $D : \mathcal{P}_3(x) \rightarrow \mathcal{P}_3(x)$ be given by the derivative map. Is this function invertible?

The function is not invertible, since it has non-trivial kernel. We can also see this by writing down the matrix relative to the obvious basis: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since there is a row of

all zeroes, the rows are not linearly independent, so the matrix is not invertible.

Let's tweak things a bit. Let $Q = \{ax + bx^2 + cx^3\} \subset \mathcal{P}_3(x)$, and let $D : Q \rightarrow \mathcal{P}_2(x)$ be given by the derivative. Then if we let $E = \{x, x^2, x^3\}$, $F = \{1, x, x^2\}$ be bases for the

domain and codomain, we see the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ whose inverse is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Thus

this function is invertible; in fact we see the inverse sends $a + bx + cx^2 \mapsto ax + bx^2/2 + cx^3/3$, which you should recognize as an integral.

As a final coda, we'll note that every vector space is isomorphic to itself. This fact will be really important when we want to change coordinate systems.

Proposition 4.59. *Let V be a vector space. Then $V \cong V$. In particular, the identity map Id_V defined by $Id_V(\mathbf{v}) = \mathbf{v}$ is an isomorphism from V to V .*

But two isomorphic vector spaces can have more than one isomorphism between them. In fact, any non-trivial vector space has infinitely many isomorphisms from itself to itself, and these isomorphisms are extremely useful.

Proposition 4.60. *Let U, V be vector spaces, let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for U , and $L : U \rightarrow V$ be a linear map. Then $L(E)$ spans $L(U)$, and L is an isomorphism if and only if $L(E)$ is a basis for V .*

Corollary 4.61. *Let U, V be vector spaces, with $\dim U = \dim V$. Then $U \cong V$.*

Proof. Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for U , and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ a basis for V ; we know they have the same number of elements since $\dim U = \dim V$. Define a linear map $L : U \rightarrow V$ by linearly extending $L(\mathbf{e}_i) = \mathbf{f}_i$; that is, we define

$$L(a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n) = a_1\mathbf{f}_1 + \dots + a_n\mathbf{f}_n.$$

This map sends a basis to a basis, and thus is an isomorphism. □

Corollary 4.62. *A linear map from V to V is an isomorphism if and only if it sends a basis to a basis.*