Common Notation

Symbol	Meaning	Reference
$\mathbb{R}$	the set of real numbers	1.1
$\in$	is an element of	1.1
$\mathbb{R}^n$	the set of ordered n-tuples	1.1
$\{a, b, d\}$	a set containing a, b, and d	1.1
$\{3x : x \in \mathbb{R}\}$	the set of all $3x$ such that $x \in \mathbb{R}$	1.1
$\mathbb{R}^2$	the set of ordered pairs of real numbers; the Cartesian plane	1.1
$\subseteq$	is a subset of	1.1
$\varnothing$	the empty set	1.1
M_n	Set of (square) $n \times n$ matrices	1.2
A^T	Transpose of A	1.4.3
I_n	Identity matrix in M_n	1.5
$\vec{0}$ or $\mathbf{0}$	the zero vector	1.6, 2.1
$N(A)$ or $\ker(A)$	Nullspace or kernel of matrix A	1.6
$\overrightarrow{AB}$	the vector from the point A to the point B	2.1
O	the point at the origin	2.1
$\vec{v}$ or $\mathbf{v}$	a vector	2.1
$\mathbb{R}^3$	Euclidean threespace	2.2
V	vector space	2.3
$\mathcal{P}(x)$	space of polynomials in x	2.3
$\mathcal{F}(\mathbb{R}, \mathbb{R})$	the space of functions from $\mathbb{R}$ to $\mathbb{R}$	2.3
$\mathbb{Z}$	the set of integers	2.3
$\cup$	union	3.1
WLOG	Without Loss of Generality	3.2
$\exists$	There exists	
$\mathbf{e}_i$ or $\vec{e}_i$	Standard basis vectors for $\mathbb{R}^n$	3.3

Symbol	**Meaning**	**Reference**
$\cong$	Is isomorphic to	4.4
$\sim$	Is similar to	6.2
$\mathbf{u} \cdot \mathbf{v}$	dot product of $\mathbf{u}$ and $\mathbf{v}$	7.1
$\lVert\mathbf{v}\rVert$	magnitude of $\mathbf{v}$	7.1
$d(\mathbf{x}, \mathbf{y})$	distance between $\mathbf{x}$ and $\mathbf{y}$	7.1
$\mathrm{proj}_{\mathbf{v}}\mathbf{u}$	The projection of $\mathbf{u}$ onto $\mathbf{v}$	7.1
$\langle \mathbf{u}, \mathbf{v} \rangle$	The inner product of $\mathbf{u}$ and $\mathbf{v}$	7.2
$U^\perp$	Orthogonal complement to U	7.4
$\mathbf{v}_U, \mathbf{v}_{U^\perp}$	Orthogonal decomposition	7.4
λ	Eigenvalue of an operator	5.1
E_λ	Eigenspace corresponding to the eigenvalue λ	5.1
$\det A$	Determinant of A	5.2
M_{ij}	The i, j minor matrix of a matrix A	5.2.1
A_{ij}	The i, j cofactor of a matrix A	5.2.1
$\chi_A(\lambda)$	Characteristic polynomial of A	5.3
$\mathrm{Tr}(A)$	Trace of A	6.3

Contents

Introduction: Changing Perspectives

In this course we want to study "high-dimensional spaces" and "vectors". That's not very specific, though, until we explain exactly what we mean by those things.

An important idea of this course is that it is helpful to study the same things from more than one perspective; sometimes a question that is difficult from one perspective is easy from another, so the ability to have multiple viewpoints and translate between them is extremely useful.

In this course we will take three different perspectives, which I am calling "geometric", "algebraic", and "formal". The first involves spatial reasoning and pictures; the second involves arithmetic and algebraic computations; the third involves formal definitions and properties. The formal perspective is the most abstract and sometimes the most confusing, but often the most fruitful: the formal perspective allows us to take problems that don't look like they involve anything we would call "vectors", and apply the techniques of linear algebra to them anyway.

A common definition of a vector is "something that has size and direction." This is a *geometric* viewpoint, since it calls to mind a picture. We can also view it from an *algebraic* point of view by giving it a set of coordinates. For instance, we can specify a two-dimensional vector by giving a pair of real numbers (x, y), which tells us where the vector points from the origin at $(0, 0)$. From the formal perspective we just have "a vector", which must satisfy certain conditions we'll state later.

In the table below I have several concepts, and ways of thinking about them in each perspective. It's fine if you don't know what some of these things mean, especially in the "formal" column; if you knew all of this already you wouldn't need to take this course.

Geometric	Algebraic	Formal
size and direction	n-tuples	vectors
consecutive motion	pointwise addition	vector addition
stretching, rotations, reflections	matrices	linear functions
number of independent directions	number of coordinates	dimension
plane	system of linear equations	subspace
angle	dot product	inner product
Length	magnitude	norm

1 Systems of Linear Equations

We're going to start this course with a very concrete, very algebraic problem: solving equations. As the course progresses, we will see how this problem relates to geometric and formal ideas. We will bring in ideas from geometric and formal perspectives to help us approach this problem, and see how we can use our equation-solving techniques to answer questions that arise in geometric and formal settings.

1.1 Basics of Linear Equations

A *linear equation* is an equation of the form

$$a_1x_1 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, \dots, a_n$, and b are all real numbers, and $x_1, \dots, x_n$ are *unknowns* or *variables*. (We might write $a_1, \dots, a_n, b \in \mathbb{R}$; the symbol $\mathbb{R}$ stands for the real numbers, and the symbol $\in$ means "is an element of" or just "in"). We say that this equation has n unknowns.

A *system of linear equations* is a system of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

with the a_{ij} and b_is all real numbers. We say this is a system of m equations in n unknowns.

Importantly, these equations are restricted to be relatively simple. In each equation we multiply each variable by some constant real number, add them together, and set that equal to some constant real number. We aren't allowed to multiply variables together, or do anything else fancy with them. This means the equations can't get too complicated, and are relatively easy to work with.

Example 1.1. A system of two linear equations in two variables is

$$\begin{aligned} 2x + y &= 3 \\ x + 5y &= -3. \end{aligned}$$

A system of two equations in three variables is

$$\begin{aligned} 5x + 2y + z &= 7 \\ 3x + 2y + z &= 6. \end{aligned}$$

A system of three equations in one variable is

$$\begin{aligned} 3x &= 3 \\ 5x &= 5 \\ x &= 2. \end{aligned}$$

We want to find *solutions* to this system of equations. Since there are n variables, a solution much be a list of n real numbers. We write $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$ for the set of ordered lists of n real numbers. (We sometimes call these "ordered n-tuples" or "vectors"). Thus $\mathbb{R}^1 = \mathbb{R}$ is just the set of real numbers; $\mathbb{R}^2$ is the set of ordered pairs that makes up the Cartesian plane.

An element $(x_1, \dots, x_n) \in \mathbb{R}^n$ is a *solution* to a system of linear equations if all of the equalities hold for that collection of x_i. The *solution set* of a system of linear equations is the set of all solutions, and we say two systems are equivalent if they have the same set of solutions.

Example 1.2. The system

$$\begin{aligned} 2x + y &= 3 \\ x + 5y &= -3 \end{aligned}$$

has $(2, -1) \in \mathbb{R}^2$ as a solution. We will see later that this is the only solution, and thus the set of solutions is $\{(2, -1)\}$.

The system

$$\begin{aligned} 4x + 2y + 2z &= 8 \\ 3x + 2y + z &= 6 \end{aligned}$$

has $(1, 1, 1)$ as a solution. This is not the only solution; in fact, the set of solutions is $\{(x, 2-x, 2-x) : x \in \mathbb{R}\}$. (This means that for each real number x, the ordered triple $(x, 2-x, 2-x)$ is a solution to our system). We say this is a *subset* of $\mathbb{R}^3$, since it is a collection of elements of $\mathbb{R}^3$, and write $\{(x, 2-x, 2-x) : x \in \mathbb{R}\} \subset \mathbb{R}^3$.

The system

$$\begin{aligned} 3x &= 3 \\ 5x &= 5 \\ x &= 2 \end{aligned}$$

clearly has no solutions, since the first equation implies that $x = 1$ but the third equation implies that $x = 2$. Thus the set of solutions is the *empty set* $\{\} = \varnothing$.

We say that two systems of equations are *equivalent* if they have the same set of solutions. Thus the process of solving a system of equations is mostly the process of converting a system into an equivalent system that is simpler.

There are three basic operations we can perform on a system of equations to get an equivalent system:

1. We can write the equations in a different order.
2. We can multiply any equation by a nonzero scalar.
3. We can add a multiple of one equation to another.

All three of these operations are guaranteed not to change the solution set; proving this is a reasonable exercise. Our goal now is to find an efficient way to use these rules to get a useful solution to our system.

Example 1.3. The system

$$\begin{aligned} 2x + y &= 3 \\ x + 5y &= -3 \end{aligned}$$

is equivalent to

$$\begin{aligned} 2x + y &= 3 \\ -2x + -10y &= 6 \end{aligned}$$

and then

$$\begin{aligned} 0x + -9y &= 9 \\ -2x + -10y &= 6 \end{aligned}$$

then

$$\begin{aligned} 0x + y &= -1 \\ -2x + -10y &= 6 \end{aligned}$$

$$\begin{aligned} 0x + y &= 1 \\ -2x + 0y &= -4 \end{aligned}$$

$$0x + y = 1$$
$$x + 0y = 2$$

which give us our solution of $x = 2, y = 1$ or $(x, y) = (2, 1)$.

This takes up a really awkward amount of space on the page, though, and we'd like to find a better and more systematic way of approaching this process.

Remark 1.4. There's another possible approach to solving these systems, called the method of substitution. We could observe that if $2x + y = 3$ then $y = 3 - 2x$, and substitute that into our other equation to give

$$\begin{aligned} x + 5(3 - 2x) &= -3 \\ 15 - 9x &= -3 \\ 9x &= 18 \\ x &= 2 \end{aligned}$$

and from here we can see that $y = 3 - 2(2) = -1$.

This is often much simpler to do in your head for small systems. But it scales up really poorly to systems with more than two or three equations and variables, so we'll want to learn something more effective.

1.2 The matrix of a system

Looking at a system of linear equations, we notice that it can be described by an array of real numbers. These numbers are naturally laid out in a rectangular grid, so we wnat to find an efficient way to represent them.

Definition 1.5. A *(real) matrix* is a rectangular array of (real) numbers. A matrix with m rows and n columns is *a* $m \times n$ *matrix*, and we notate the set of all such matrices by $M_{m\times n}$.

A $m \times n$ matrix is *square* if $m = n$, that is, it has the same number of rows as columns. We will sometimes represent the set of $n \times n$ square matrices by M_n.

We will generally describe the elements of a matrix with the notation

$$(a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

We can now take the information from a system of linear equations and encode it in a matrix. Right now, we will just use this as a convenient notational shortcut; we will see later on in the course that this has a number of theoretical and practical advantages.

Definition 1.6. The *coefficient matrix* of a system of linear equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and the *augmented coefficient matrix* is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Example 1.7. Suppose we have a system

$$\begin{aligned} 4x + 2y + 2z &= 8 \\ 3x + 2y + z &= 6. \end{aligned}$$

Then the coefficient matrix is

$$\begin{bmatrix} 4 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

and the augmented coefficient matrix is

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 8 \\ 3 & 2 & 1 & 6. \end{array}\right]$$

Earlier we listed three operations we can perform on a system of equations without changing the solution set: we can reorder the equations, multiply an equation by a nonzero scalar, or add a multiple of one equation to another. We can do analogous things to the coefficient matrix.

Definition 1.8. The three *elementary row operations* on a matrix are

I Interchange two rows.

II Multiply a row by a nonzero real number.

III Replace a row by its sum with a multiple of another row.

Example 1.9. What can we do with our previous matrix? We can

$$\begin{bmatrix} 4 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} \overset{I}{\to} \begin{bmatrix} 3 & 2 & 1 \\ 4 & 2 & 2 \end{bmatrix} \overset{II}{\to} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \overset{III}{\to} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

So how do we use this to solve a system of equations? The basic idea is to remove variables from successive equations until we get one equation that contains only one variable—at which point we can substitute for that variable, and then the others. To do that with this matrix, we have

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 8 \\ 3 & 2 & 1 & 6 \end{array}\right] \overset{III}{\to} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 6 \end{array}\right] \overset{III}{\to} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & 0 \end{array}\right] \overset{II}{\to} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right].$$

What does this tell us? That our system of equations is equivalent to the system

$$\begin{aligned} x + z &= 2 \\ y - z &= 0. \end{aligned}$$

This gives us the answer I stated earlier: $z = 2 - x$ and $y = z = 2 - x$.

Example 1.10. Solve the system of equations

$$\begin{aligned} x + 2y + z &= 3 \\ 3x - y - 3z &= -1 \\ 2x + 3y + z &= 4. \end{aligned}$$

This system has augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \overset{III}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 2 & 3 & 1 & 4 \end{array}\right] \overset{III}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right]$$

$$\overset{II}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 1 & 1 & 2 \end{array}\right] \overset{I}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -7 & -6 & -10 \end{array}\right] \overset{III}{\to} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

which gives us the system

$$\begin{aligned} x + 2y + z &= 3 \\ y + z &= 2 \\ z &= 4. \end{aligned}$$

The last equation tells us $z = 4$, which then gives $y = -2$ and $x = 3$. We can check that this solves the system.

1.3 Row Echelon Form

We want to solve systems of linear equations, using these matrix operations. We want to be somewhat more concrete about our goals: what exactly would it look like for a system to be solved?

Definition 1.11. A matrix is in *row echelon form* if

- Every row containing nonzero elements is above every row containing only zeroes; and
- The first (leftmost) nonzero entry of each row is to the right of the first nonzero entry of the above row.

Remark 1.12. Some people require the first nonzero entry in each nonzero row to be 1. This is really a matter of taste and doesn't matter much, but you should do it to be safe; it's an easy extra step to take by simply dividing each row by its leading coefficient.

Example 1.13. The following matrices are all in Row Echelon Form:

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 3 & -1 & 4 \\ 0 & 0 & -2 & 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 & 3 & 2 & 8 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -7 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & 3 \\ 0 & 0 & 7 \end{bmatrix}.$$

The following matrices are not in Row Echelon Form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 5 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 1.14. The process of using elementary row operations to transform a system into row echelon form is *Gaussian elimination.*

A system of equations sometimes has a solution, but does not always. We say a system is *inconsistent* if there is no solution; we say a system is *consistent* if there is at least one solution.

Example 1.15. Consider the system of equations given by

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ -1x_1 + -1x_2 + x_5 &= -1 \\ -2x_1 + -2x_2 + 3x_5 &= 1 \\ x_3 + x_4 + 3x_5 &= -1 \\ x_1 + x_2 + 2x_3 + 2x_4 + 4x_5 &= 1. \end{aligned}$$

This translates into the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right].$$

We see that the final two equations are now $0 = -4$ and $0 = -3$,s othe system is inconsistent.

Example 1.16. Let's look at another system that is almost the same.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ -1x_1 + -1x_2 + x_5 &= -1 \\ -2x_1 + -2x_2 + 3x_5 &= 1 \\ x_3 + x_4 + 3x_5 &= 3 \\ x_1 + x_2 + 2x_3 + 2x_4 + 4x_5 &= 4. \end{aligned}$$

This translates into the augmented matrix

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We see this system is now consistent. Our three equations are

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1 \qquad x_3 + x_4 + 2x_5 = 0 \qquad x_5 = 3.$$

Via back-substitution we see that we have

$$x_5 = 3 \qquad x_3 + x_4 = -6 \qquad x_1 + x_2 = 4.$$

Thus we could say the set of solutions is $\{(\alpha, 4 - \alpha, \beta, -6 - \beta, 3)\} \subseteq \mathbb{R}^5$.

What we were just doing definitely worked, but even after we finished transforming the matrix we still needed to do some more work. So we'd like to reduce the matrix even further until we can just read the answer off from it.

Definition 1.17. A matrix is in *reduced row echelon form* if it is in row echelon form, and the first nonzero entry in each row is the only entry in its column.

This means that we will have some number of columns that each have a bunch of zeroes and one 1. Other than that we may or may not have more columns, which can contain basically anything; we've used up all our degrees of freedom to fix those columns that contain the leading term of some row.

Note that the columns we have fixed are not necessarily the first columns, as the next example shows.

Example 1.18. The following matrices are all in reduced Row Echelon Form:

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 17 & 0 & 2 & 8 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following matrices are not in reduced Row Echelon Form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 15 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 1.19. Let's solve the following system by putting the matrix in reduced row echelon form.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

We have

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

From this we can read off the solution $x_1 + x_2 + x_3 = 1, x_4 = 2, x_5 = -1$. Thus the set of solutions is $\{(1 - \alpha - \beta, \alpha, \beta, 2, -1)\}$.

We say some systems of equations are "overdetermined", which means that there are more equations than varaibles. Overdetermined equations are "usually" inconsistent, but not always—they can be consistent when some of the equations are redundant.

Example 1.20. The system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned}$$

gives the matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 3/5 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1/10 \\ 0 & 1 & 0 & -3/10 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives us the solution $x_1 = 1/10, x_2 = -3/10, x_3 = 3/2$, which you can go back and check solves the original system.

This overdetermined system does have a solution, but only because two of the equations were redundant, as we could see in the second matrix where two lines are identical. In fact we can go back to the original set of equations, and see that if we add two times the first equation to the second equation, we get the third—which is the redundancy.

Other systems of equations are "underdetermined", which means there are more variables than equations. These systems are usually but not always consistent.

Example 1.21. Let's consider the system

$$\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 + x_2 - 2x_3 - x_4 &= 0. \end{aligned}$$

This gives us the matrix

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & 1 & -2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 3 & -4 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 3 & -4 & 5 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

We see that we can't "simplify" the fourth column in any way; we don't have any degrees of freedom after we fix the first three columns. This means that we can pick x_4 to be anything we want, and the other variables are given by $x_1 - x_4 = 0, x_2 - 3x_4 = 0, x_3 + x_4 = 0$. Thus the set of solutions is $\{(\alpha, 3\alpha, -\alpha, \alpha)\}$.

Remark 1.22. A system of any size can be either consistent or inconsistent. $0 = 1$ is an inconsistent system with one equation, and

$$x_1 + \cdots + x_{100} = 0$$
$$x_1 + \cdots + x_{100} = 1$$

is an inconsistent system with a hundred variables and only two equations. In contrast,

$$x_1 = 1$$
$$x_1 = 1$$
$$\vdots$$
$$x_1 = 1$$

has only one variable, and many equations, and is still consistent.

1.4 Matrix Algebra

So far we've treated matrices as just being a convenient way to write down a bunch of numbers. But matrices are interesting mathematical objects in their own right, and we can do a lot of useful calculations with them.

1.4.1 Simple Operations

We want to start with a couple of simple operations. Neither of these operations really depend on the structure of the matrix; they treat the matrix as a list of numbers.

Definition 1.23. If $A = (a_{ij})$ is an $m \times n$ matrix, and $r \in \mathbb{R}$ is a real number, then we can multiply each entry of the matrix A by the real number R. This is called *scalar multiplication* and we say that r is a *scalar*.

$$rA = (ra_{ij}) = \begin{bmatrix} ra_{11} & ra_{12} & \dots & ra_{1n} \\ ra_{21} & ra_{22} & \dots & ra_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ra_{m1} & ra_{m2} & \dots & ra_{mn} \end{bmatrix}.$$

Definition 1.24. If $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices, we can add the two matrices by adding each individual pair of coordinates together.

$$A + B = (a_{ij} + b_{ij}) = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}.$$

Example 1.25.

$$3 \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ -3 & 12 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & 3 \\ -2 & 5 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 7 & 5 \\ 1 & -6 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 8 \\ -1 & -1 & 3 \end{bmatrix}$$

1.4.2 Matrix Multiplication

Definition 1.26. If $A \in M_{\ell \times m}$ and $B \in M_{m \times n}$, then there is a matrix $AB \in M_{\ell \times n}$ whose ij element is

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

If you're familiar with the dot product, you can think that the ij element of AB is the dot product of the ith row of A with the jth column of b.

Note that A and B don't have to have the same dimension! Instead, A has the same number of columns that B has rows. The new matrix will have the same number of rows as A and the same number of columns as B.

Example 1.27.

$$\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\begin{bmatrix}5 & -1\\ 3 & 2\end{bmatrix} = \begin{bmatrix}1\cdot 5+3\cdot 3 & 1\cdot(-1)+3\cdot 2\\ 2\cdot 5+4\cdot 3 & 2\cdot(-1)+4\cdot 2\end{bmatrix} = \begin{bmatrix}14 & 5\\ 22 & 6\end{bmatrix}$$

$$\begin{bmatrix}4 & 6\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 1 & 5\\ 4 & 1 & 6\end{bmatrix} = \begin{bmatrix}4\cdot 3+6\cdot 4 & 4\cdot 1+6\cdot 1 & 4\cdot 5+6\cdot 6\\ 2\cdot 3+1\cdot 4 & 2\cdot 1+1\cdot 1 & 2\cdot 5+1\cdot 6\end{bmatrix} = \begin{bmatrix}36 & 10 & 56\\ 10 & 3 & 16\end{bmatrix}.$$

Matrix multiplication is *associative*, by which we mean that $(AB)C = A(BC)$.

Matrix multiplication is not commutative: in general, it's not even the case that AB and BA both make sense. If $A \in M_{3\times 4}$ and $B \in M_{4\times 2}$ then AB is a 3×2 matrix, but BA isn't a thing we can compute. But even if AB and BA are both well-defined, they are not equal.

Example 1.28.

$$\begin{bmatrix}3 & 5 & 1\\ -2 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\\ 4 & 1\end{bmatrix} = \begin{bmatrix}3\cdot 2+5\cdot 1+1\cdot 4 & 3\cdot 1+5\cdot 3+1\cdot 1\\ -2\cdot 2+0\cdot 1+2\cdot 4 & -2\cdot 1+0\cdot 3+2\cdot 1\end{bmatrix} = \begin{bmatrix}15 & 19\\ 4 & 0\end{bmatrix}$$

$$\begin{bmatrix}2 & 1\\ 1 & 3\\ 4 & 1\end{bmatrix}\begin{bmatrix}3 & 5 & 1\\ -2 & 0 & 2\end{bmatrix} = \begin{bmatrix}2\cdot 3+1\cdot(-2) & 2\cdot 5+1\cdot 0 & 2\cdot 1+1\cdot 2\\ 1\cdot 3+3\cdot(-2) & 1\cdot 5+3\cdot 0 & 1\cdot 1+3\cdot 2\\ 4\cdot 3+1\cdot(-2) & 4\cdot 5+1\cdot 0 & 4\cdot 1+1\cdot 2\end{bmatrix} = \begin{bmatrix}4 & 10 & 4\\ -3 & 5 & 7\\ 10 & 20 & 6\end{bmatrix}.$$

Particularly nice things happen when our matrices are square. Any time we have two $n \times n$ matrices we can multiply them by each other in either order (though we will still get different things each way!).

Example 1.29.

$$\begin{bmatrix}4 & 1\\ -3 & 5\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & -2\end{bmatrix} = \begin{bmatrix}-3 & 2\\ 8 & -13\end{bmatrix}$$

$$\begin{bmatrix}-1 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}4 & 1\\ -3 & 5\end{bmatrix} = \begin{bmatrix}-7 & 4\\ 10 & -9\end{bmatrix}.$$

However, matrix multiplication does satisfy the *distributive* and *associative* properties.

Fact 1.30. *If $A \in M_{\ell\times m}$ and $B, C \in M_{m\times n}$ then $A(B + C) = AB + AC$.*

If $A \in M_{\ell\times m}, B \in M_{m\times n}, C \in M_{n\times p}$ then $(AB)C = A(BC)$.

Example 1.31. Let

$$A = \begin{bmatrix}4 & 1\\ -3 & 5\end{bmatrix} \qquad B = \begin{bmatrix}-1 & 1\\ 1 & -2\end{bmatrix} \qquad C = \begin{bmatrix}3 & 2\\ 1 & -5\end{bmatrix}.$$

Then we have

$$AB = \begin{bmatrix} -3 & 2 \\ 8 & -13 \end{bmatrix} \qquad AC = \begin{bmatrix} 13 & 3 \\ -4 & -31 \end{bmatrix} \qquad AB + AC = \begin{bmatrix} 10 & 5 \\ 4 & -44 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 2 & 3 \\ 2 & -7 \end{bmatrix} \qquad A(B+C) = \begin{bmatrix} 10 & 5 \\ 4 & -44 \end{bmatrix}.$$

Thus we see $AB + AC = A(B + C)$.

We can similarly compute

$$AB = \begin{bmatrix} -3 & 2 \\ 8 & -13 \end{bmatrix} \qquad (AB)C = \begin{bmatrix} -3 & 2 \\ 8 & -13 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} -7 & -16 \\ 11 & 81 \end{bmatrix}$$

$$BC = \begin{bmatrix} -2 & -7 \\ 1 & 12 \end{bmatrix} \qquad A(BC) = \begin{bmatrix} 4 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} -2 & -7 \\ 1 & 12 \end{bmatrix} = \begin{bmatrix} -7 & -16 \\ 11 & 81 \end{bmatrix}$$

1.4.3 Transposes

Definition 1.32. If A is a $m \times n$ matrix, then we can form a $n \times m$ matrix B by flipping A across its diagonal, so that $b_{ij} = a_{ji}$. We say that B is the *transpose* of A, and write $B = A^T$.

If $A = A^T$ we say that A is *symmetric*. (Symmetric matrices must always be square).

Example 1.33.

$$\text{If } A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 4 & 2 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}.$$

$$\text{If } B = \begin{bmatrix} 5 & 3 \\ 3 & -2 \end{bmatrix} \text{ then } B^T = \begin{bmatrix} 5 & 3 \\ 3 & -2 \end{bmatrix}$$

and thus B is symmetric.

Fact 1.34.

- $(A^T)^T = A$.
- $(A + B)^T = A^T + B^T$.
- $(rA)^T = rA^T$.
- *If* $A \in M_{\ell \times m}$ *and* $B \in M_{m \times n}$ *then* $(AB)^T = B^T A^T$.

1.4.4 Matrices and Systems of Equations

We will do a lot with matrices in the future (a linear algebra class that doesn't cover general vector spaces is often called a matrix algebra class). In the current context we mostly want it to make it easier to talk about systems of equations.

Let

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

be a system of linear equations. Then $A = (a_{ij}) \in M_{m\times n}$ is its coefficient matrix, and $\mathbf{b} = (b_1, \dots, b_m)$ is an element of $\mathbb{R}^m$, but we can also think of it as a $m \times 1$ matrix $b = [b_1, \dots, b_m]^T$. If we take $\mathbf{x} = [x_1, \dots, x_n]^T$ to be a $n \times 1$ matrix, we can rewrite our linear system as the equation

$$A\mathbf{x} = \mathbf{b},$$

which is certainly much easier to write down.

Example 1.35. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{b} = [4, 6]^T$, then the equation $A\mathbf{x} = \mathbf{b}$ is

$$\begin{aligned} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ \begin{bmatrix} x + 3y \\ 2x + 4y \end{bmatrix} &= \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ x + 3y &= 4 \\ 2x + 4y &= 6 \end{aligned}$$

1.5 The identity matrix and matrix inverses

We just saw that any system of linear equations can be written $A\mathbf{x} = \mathbf{b}$, which reminds us of the single-variable linear equation $ax = b$. In the single-variable case we can just divide both sides of the equation by a, as long as $a \neq 0$; it would be nice if we can do the same thing for any system of linear equations.

But what does it mean to divide by a matrix? When we define division, we often start by understanding reciprocals $\frac{1}{a}$. So we start by asking what matrix is the equivalent of the number 1.

Definition 1.36. For any n we define the *identity matrix* to be $I_n \in M_n$ to have a 1 on every diagonal entry, and a zero everywhere else. For example,

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If $A \in M_n$ then $I_n A = A = AI_n$. Thus it is a *multiplicative identity* in the ring of $n \times n$ matrices.

The identity matrix is symmetric (that is, $I_n^T = I_n$).

Now we want to define multiplicative inverses, the equivalent of reciprocals. The definition is not difficult to invent:

Definition 1.37. Let A and B be $n \times n$ matrices, such that $AB = I_n = BA$. Then we say that B is the *inverse* (or *multiplicative inverse*) of A, and write $B = A^{-1}$.

If such a matrix exists, we say that A is *invertible* or *nonsingular*. If no such matrix exists, we say that A is *singular*.

Example 1.38. The identity matrix I_n is its own inverse, and thus invertible.

The matrices

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/10 & 2/5 \\ 3/10 & -1/5 \end{bmatrix}$$

are inverses to each other, as you can check.

Example 1.39. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has no inverse, since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

won't be the identity for any a, b, c, d. Thus this matrix is singular.

Remark 1.40. If $AB = I_n$ then $BA = I_n$. This isn't really trivial but we won't prove it.

As the last example shows, finding the inverse to a matrix is a matter of solving a big pile of linear equations at the same time (one for each coefficient of the inverse matrix). Fortunately, we just got good at solving linear equations. Even more fortunately, there's an easy way to organize the work for these problems.

Proposition 1.41. *Let A be a $n \times n$ matrix. Then if we form the augmented matrix $\begin{bmatrix} A & I_n \end{bmatrix}$, then A is invertible if and only if the reduced row echelon form of this augmented matrix is $\begin{bmatrix} I_n & B \end{bmatrix}$ for some matrix B, and furthermore $B = A^{-1}$.*

Proof. Let X be a $n \times n$ matrix of unknowns, and set up the system of equations implied by $AX = I_n$. This will be the same set of equations we are solving with this row reduction, and thus a matrix X exists if and only if this system has a solution, which happens if and only if the reduced row echelon form of $\begin{bmatrix} A & I_n \end{bmatrix}$ has no all-zero rows on the A side. □

Example 1.42. Let's find an inverse for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

We form and reduce the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$. We can check this by multiplying the matrices back together.

Example 1.43. Find the inverse of $B = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 6 \\ -3 & 0 & -10 \end{bmatrix}$.

We form and reduce the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 6 & 0 & 1 & 0 \\ -3 & 0 & -10 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3/2 & 0 & 1/2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 0 & -2 \\ 0 & 1 & 2 & -4 & 1 & -1 \\ 0 & 0 & 1 & 3/2 & 0 & 1/2 \end{array}\right].$$

Thus $B^{-1} = \begin{bmatrix} -5 & 0 & -2 \\ -4 & 1 & -1 \\ 3/2 & 0 & 1/2 \end{bmatrix}$.

Example 1.44. What happens if we try to find an inverse for $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? We start with

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

but then there is no way to make the left-side block of the matrix into the identity I_2. Thus this matrix C is not invertible.

There are many more interesting properties of inverse matrices we'd like to discuss, but we don't have the tools to explain them properly yet. We will be returning to the properties of matrices throughout the course as we develop more techniques and vocabulary.

1.6 Homogeneous systems and subspaces

There's one particular category of systems of linear equations that's especially important to us, and will lead into the main subject matter of the course.

Definition 1.45. The $n \times 1$ matrix $\mathbf{0} = [0, \ldots, 0]^T$ whose entries are all zero is called the *zero vector.*

A system of linear equations $A\mathbf{x} = \mathbf{b}$ is called *homogeneous* if $\mathbf{b} = \mathbf{0}$, that is, if the constant term in each equation is zero. Otherwise, it is *non-homogeneous.*

It's pretty clear that every homogeneous system has at least one solution: the solution where every variable is equal to zero. It may have many more solutions than that.

Definition 1.46. For a given matrix A, the subspace of solutions to the equation $A\mathbf{x} = \mathbf{0}$ is called the *nullspace* $N(A)$ or the *kernel* $\ker(A)$ of the matrix A.

Example 1.47. Find the null space of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$.

We row reduce the matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

We see that x_3 and x_4 are fixed variables, and x_1, x_2 are determined by x_3 and x_4. (You could of course do this the other way around). Then we have $x_1 = x_3 - x_4$ and $x_2 = x_4 - 2x_3$.

Thus $N(A) = \{(\alpha - \beta, \beta - 2\alpha, \alpha, \beta)\} = \{\alpha(1, -2, 1, 0) + \beta(-1, 1, 0, 1)\}$.

Remark 1.48. It's not too hard to see that a square matrix A is invertible if and only if $N(A) = \{\mathbf{0}\}$. If the matrix is invertible, then row-reducing it gets to be the identity matrix—and so the solution to the associated homogeneous system is just $\mathbf{0}$. Conversely, if the only solution is $\mathbf{0}$ then you must not have any rows of all zeros in the reduced form of your matrix, so it's invertible.

We can see that if we add together two solutions to this system of equations, we will get another. In fact, this must be true of any homogeneous system.

Proposition 1.49 (Homogeneity). *Suppose $A\mathbf{x} = \mathbf{0}$ is a homogeneous sytem of linear equations. Then:*

1. *$\mathbf{0}$ is a solution to the system.*
2. *If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to this system, then $\mathbf{x}_1 + \mathbf{x}_2$ is a solution.*
3. *If $\mathbf{x}$ is a solution to this system, and r is a real number, then $r\mathbf{x}$ is a solution.*

Remark 1.50. We can rephrase this result: for any matrix A, we have

1. $\mathbf{0} \in N(A)$
2. If $\mathbf{x}_1, \mathbf{x}_2 \in N(A)$ then $\mathbf{x}_1 + \mathbf{x}_1 \in N(A)$
3. If $r \in \mathbb{R}$ and $\mathbf{x} \in N(A)$ then $r\mathbf{x} \in N(A)$.

This says exactly the same thing, but puts the emphasis on the matrix A rather than on the equation $A\mathbf{x} = \mathbf{0}$.

Proof. 1. Calculation confirms that $A\mathbf{0} = \mathbf{0}$.

2. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = 0$, so we have

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{x}_1 + \mathbf{x}_2$ is a solution.

3. If $\mathbf{x}$ is a solution and $r \in \mathbb{R}$, then

$$A(r\mathbf{x}) = rA\mathbf{x} = r\mathbf{0} = \mathbf{0}.$$

Thus $r\mathbf{x}$ is a solution.

□

In contrast, the set of solutions to a non-homogeneous system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$ never has these nice properties.

1. The zero vector is never a solution, since $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$.
2. Adding two solutions doesn't give you another solution: $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$.
3. Multiplying a solution by a scalar doesn't give another solution: $Ar\mathbf{x} = r\mathbf{b} \neq \mathbf{b}$ unless $r = 1$.

So there's something special about homogeneous systems, which we will discuss in more detail in 2.3.

But even though the set of solutions to a non-homogeneous system doesn't have the nice properties of proposition 1.49, we can still say a lot about what it looks like.

Proposition 1.51. *Suppose $A\mathbf{x} = \mathbf{b}$ is a non-homogeneous linear system.*

If $U = N(A))$ and $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{b}$, then the set of solutions to the system $A\mathbf{x} = \mathbf{b}$ is the set

$$N(A) + \mathbf{x}_0 = \{\mathbf{y} + \mathbf{x}_0 : \mathbf{y} \in N(A)\}.$$

Proof. We want to show that two sets are equal, so we show that each is a subset of the other.

First, suppose that $\mathbf{x}_1$ is a solution to $A\mathbf{x}_1 = \mathbf{b}$. Then we have

$$\begin{aligned} b &= A\mathbf{x}_0 \\ b &= A\mathbf{x}_1 \\ b - b &= A\mathbf{x}_1 - A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_0) \\ \mathbf{0} &= A(\mathbf{x}_1 - \mathbf{x}_0). \end{aligned}$$

Thus $\mathbf{y} = \mathbf{x}_1 - \mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$, and then $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{y}$ for some $\mathbf{y} \in U$.

Conversely, suppose $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{y}$ for some $\mathbf{y} \in U$. Then

$$A\mathbf{x}_1 = A(\mathbf{x}_0 + \mathbf{y}) = A\mathbf{x}_0 + A\mathbf{y} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Thus x_1 is a solution to $A\mathbf{x} = \mathbf{b}$. □

Remark 1.52. Notice this did not depend on the specific matricx, or even really the fact that A is a matrix at all; it only depends on the ability to distribute matrix multiplication across sums of vectors. Operations with this property are called "linear" and we will discuss them in much more detail in section 4.

Definition 1.53. Suppose $A\mathbf{x} = \mathbf{b}$ is a system of linear equations. We call the equation $A\mathbf{x} = \mathbf{0}$ the associated homogeneous system of linear equations. That is, the associated homogeneous system has the same coefficients for all the variables, but the constants are all zero.

Thus proposition 1.51 lets us understand the set of solutions to a non-homogeneous system based on the solutions to the associated homogeneous system.

Example 1.54. Let's find a set of solutions to the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 3x_2 + 4x_2 &= 9. \end{aligned}$$

Gaussian elimination gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Taking $x_3 = \alpha$ as a free variable, our solution set is $\{(\alpha, 3-2\alpha, \alpha)\} = \{(0,3,0)+\alpha(1,-2,1)\}$. Indeed, we see that this set corresponds to elements of the vector space spanned by $\{(1,-2,1)\}$, plus a specific solution $(0,3,0)$.

Alternatively, we could have solved the homogeneous system first, and seen that the solution was $x_1 - x_3 = 0, x_2 + 2x_3 = 0$, telling us that $N(A) = \{\alpha(1,-2,1)\}$. Then we just need to find a solution; to my eyes the obvious solution is $(1,1,1)$. So our theorem tells us that the solution set is $\{(1,1,1) + \alpha(1,-2,1)\}$. This may not *look* like the solution we got before, but it is in fact the same set, since $(1,1,1) = (0,3,0) + (1,-2,1)$.

Example 1.55. Now consider the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ 2x_1 + 3x_2 + 4x_2 &= 3. \end{aligned}$$

It's easy enough to see that this system has no solutions, since the sum of the first two equations should be the third.

This at first might seem concerning, since $N(A)$ is never empty. But our proposition assumed that there was at least one solution to the non-homogeneous system; when there are no solutions, the proposition doesn't actually say anything. But *if* any solution exists, proposition 1.51 tells us that the set of solutions is just the nullspace of A, plus an offset.

Example 1.56. Let's find the set of solutions to

$$\begin{aligned} x + y + z &= 0 \\ x - 2y + 2z &= 4 \\ x + 2y - z &= 2. \end{aligned}$$

We form the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 1 & 4 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

giving us the sole solution $x_1 = 4, x_2 = -2, x_3 = -2$.

If we look at the corresponding homogeneous system, we see that we can reduce the matrix to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and thus the sole solution to the homogeneous system of equations is $x_1 = x_2 = x_3 = 0$. Then every solution to our non-homogeneous system is a solution to our homogeneous system plus some vector in $\{\vec{0}\}$. Since there is only one vector in that set, there is only one solution to our system.

2 Vectors and Vector Spaces

In section 1.6 we saw that the set of solutions to a homogeneous system of linear equations has the following three properties:

1. It contains the trivial solution of all zeroes;
2. The sum of two solutions is a solution;
3. Any scalar multiple of a solution is a solution.

This set of three properties is very common and very powerful, and the fundamental subject of this course is to study sets that have that list of three properties. When thought of in this abstract way, we call such a set a "vector space".

In this section we will define vector spaces, look at a number of examples, and understand some of their fundamental properties.

However, before we proceed to this abstract idea, we should look at another very concrete example, which is important enough to give its name to the idea as a whole. Thus we will begin this section with geometry, return to algebra, and finally come up with a formal system that ties everything together.

2.1 The Cartesian Plane

We'll start by considering the "Cartesian plane", (named after the French mathematician René Descartes, who is credited with inventing the idea of putting numbered cordinates on the plane).

As probably looks familiar from high school geometry, given two points A and B in the plane, we can write $\overrightarrow{AB}$ for the vector with *initial point* A and *terminal point* B.

Since a vector is just a length and a direction, the vector is "the same" if both the initial and terminal points are shifted by the same amount. If we fix an *origin* point O, then any point A gives us a vector $\overrightarrow{OA}$. Any vector can be shifted until its initial point is O, so each vector corresponds to exactly one point. We call this *standard position*.

We represent points algebraically with pairs of real numbers, since points in the plane are determined by two coordinates. Thus we use $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ to denote the set of all ordered pairs of real numbers, giving us an algebraic description of the Cartesian plane. We define the origin O to be the "zero" point $(0, 0)$.

Definition 2.1. If $A = (x, y)$ is a point in $\mathbb{R}^2$, then we denote the vector $\overrightarrow{OA}$ by $\begin{bmatrix} x \\ y \end{bmatrix}$.

We can also denote this vector $[x, y]^T$, as we did in section 1.4.3. Poole sometimes just writes $[x, y]$, and when we don't particularly care about the geometric distinction between a point and a vector we will often write (x, y).

However, the vertical orientation is very important for a lot of calculations we will want to do, including the sort of matrix multiplication we used in section 1.4.4. Therefore we will use the vertical form when it isn't terribly inconvenient.

If we want to discuss "a vector" without specifying any coordintes, we will use a single letter, generally either boldface ($\mathbf{v}$) or with an arrow on top ($\vec{v}$).

The vector $\overrightarrow{OO}$ can't really be drawn—it's the vector with zero length—but it is very important. We call it the *zero vector* and write it as $\vec{0}$ or $\mathbf{0}$.

Example 2.2. Suppose $A = (2, 3)$ and $B = (1, 5)$. Then the vector $\overrightarrow{AB}$ has displacement in the x direction of $1 - 2 = -1$, and in the y direction of $5 - 3 = 2$. Thus it is the same as the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ which begins at $(0, 0)$ and ends at $(-1, 2)$.

If we want to take the same vector $\overrightarrow{AB}$ and put its initial point at $(-1, 2)$, then the terminal point will have x coordinate $-1 - 1 = -2$ and y-coordinate $2 + 2 = 4$, and thus be at the point $(-2, 4)$.

2.1.1 Scalar Multiplication

Geometrically, a vector is a direction and a distance. A natural question to ask is "what happens if we go in the same direction, but twice as far?" Or three times, or five times, or π times as far?

Definition 2.3. If $\mathbf{v}$ is a vector and r is a positive real number, we define *scalar multiplication* by setting $r \cdot \mathbf{v}$ to be a vector with the same direction as $\mathbf{v}$, but with its length stretched by a factor of r.

If r is a negative real number then we define $r \cdot \mathbf{v}$ to be the vector with the opposite direction from $\mathbf{v}$, and length equal to $|r|$ times the length of v.

We define $0 \cdot \mathbf{v} = \mathbf{0}$ to be the zero vector.

Remark 2.4. Notice that this means $-1 \cdot \mathbf{v}$ is a vector of the same length, but pointing in the opposite direction. So $(-1) \cdot \overrightarrow{AB} = \overrightarrow{BA}$.

Example 2.5. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Then we see that $2 \cdot \mathbf{v}$ must go twice as far in the same direction, and thus $2 \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$. Similarly, $-2 \cdot \mathbf{v} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$. Of course, we know that $0 \cdot \mathbf{v} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Looking at these examples suggests an algebraic rule for scalar multiplication:

Definition 2.6. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a vector and r is a real number, then we define *scalar multiplication* by $b \cdot \mathbf{v} = b \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} bv_1 \\ bv_2 \end{bmatrix}$. We sometimes say that scalar multiplication is given by *componentwise* multiplication.

Notice that this is exactly the scalar matrix multiplication of section 1.4.1.

Example 2.7. If $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ then $7 \cdot \mathbf{v} = \begin{bmatrix} 21 \\ 35 \end{bmatrix}$ and $\pi \cdot \mathbf{v} = \begin{bmatrix} 3\pi \\ 5\pi \end{bmatrix}$.

Remark 2.8. It is very important that scalar multiplication combines two different types of information. We have a real number r, which is a "size" without direction. We also have a vector $\mathbf{v}$ which is a magnitude and direction, and we multiply these two things together.

We cannot multiply two vectors to get another vector (outside of some very specific circumstances like the cross product). We can, of course, multiply two scalars together to get another scalar; you have been doing that since elementary school.

2.1.2 Vector Addition

Another question to ask about geometric vectors is "what happens if we go in this direction for this distance, and then once we get there, go in that direction for that distance?" In our diagram of the plane, this is represented by taking two vectors and placing them "head-to-tail".

Definition 2.9. If $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{BC}$, then we define *vector addition* by $\mathbf{v} + \mathbf{w} = \overrightarrow{AC}$.

Example 2.10. If $A = (1, 2)$, $B = (3, 1)$, $C = (5, -1)$, then we have $\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$.

Example 2.11. If $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ then we can set $A = (0, 0), B = (5, 2), C = (1, 3)$ and have $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{BC}$. Then $\mathbf{v} + \mathbf{w} = \overline{AC} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Drawing a picture every time we want to add vectors gets tedious very quickly. Fortunately, vector addition is easy algebraically: we can just do *componentwise addition.*

Definition 2.12. Algebraically, we define addition of vectors by $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$.

You can see that this gives the same result as the head-to-tail method. And again, it is the same as the matrix addition of section 1.4.1.

Remark 2.13. Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can form a parallelogram with those vectors as two of its sides. We call this the *parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$. In this case, we see that $\mathbf{u} + \mathbf{v}$ is the vector corresponding to the diagram of the parallelogram.

2.2 Threespace and $\mathbb{R}^n$

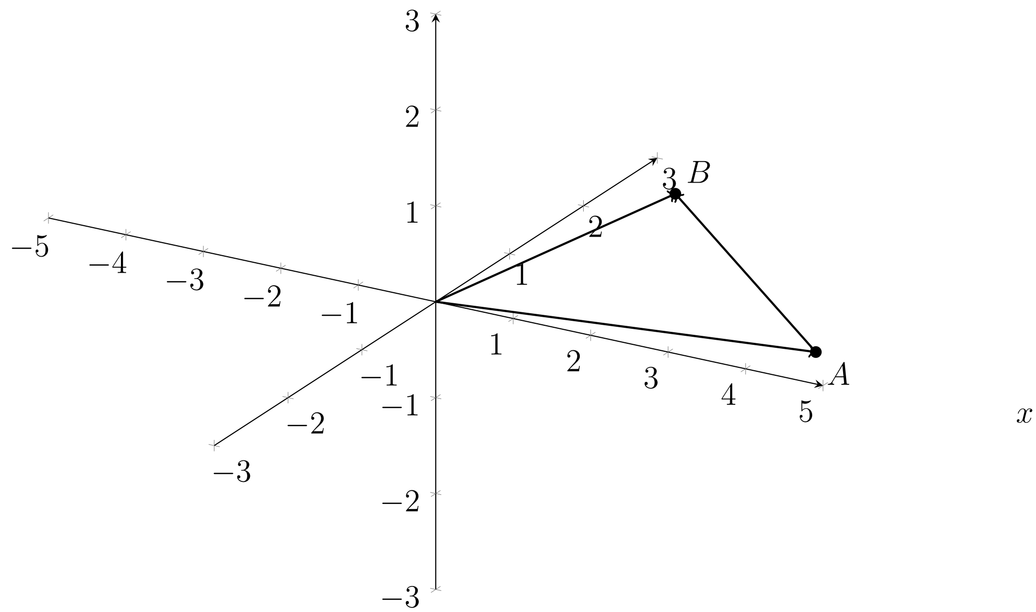
All of the work in section 2.1 took place in the "two-dimensional" plane. We can easily extend this work to three-dimensional space. Where each point in the plane requires two coordinates to express, each point in threespace requires three coordinates.

Definition 2.14. We define *Euclidean threespace* to be the three-dimensional space described by three real coordinates. We notate it $\mathbb{R}^3$. The point $(0, 0, 0)$ is called the *origin* and often notated O.

This describes familiar three-dimensional space, in which we all (apparently) live. Just as in the Cartesian plane $\mathbb{R}^2$, we can think about vectors between points.

Example 2.15. Let $A = (3, 2, -1)$ and $B = (5, -2, 3)$. Then we have

$$\overrightarrow{OA} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad \overrightarrow{OB} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}, \text{ and } \quad \overrightarrow{AB} = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}.$$



We can do vector addition and scalar multiplication as before, too.

Example 2.16. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$. Then

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}, \quad 3 \cdot \mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \text{ and } \quad (-2) \cdot \mathbf{w} = \begin{bmatrix} -8 \\ 4 \\ -6 \end{bmatrix}.$$

We have so far defined two-dimensional space and three-dimensional space. Geometrically it's hard to go farther, since most of us can't visualize a four- or five-dimensional space. (The Greeks actually argued that while you could raise a number to the second power or the third power, it made no sense to talk about 3^4 since there was no reasonable geometric interpretation. This dispute was only finally resolved in 1637 when René Descartes published a geometric method of taking two line segments, and constructing a line segment whose length was the product of the original lengths; this allowed scholars to multiply two distances and obtain a distance, resolving the philosophical concerns).

But algebraically, there's no difficulty in extending our definitions to higher dimensions and more coordinates in our vectors. (This is probably a large portion of why this course is called "linear algebra" and not "linear geometry").

Definition 2.17. We define *real n-dimensional space* to be the set of n-tuples of real numbers, $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$.

By "abuse of notation" we will also use $\mathbb{R}^n$ to refer to the set of vectors in $\mathbb{R}^n$. We define scalar multiplication and vector addition by

$$r \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \\ \vdots \\ rx_n \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Example 2.18. Let $\mathbf{v} = (1, 3, 2, 4)$ and $\mathbf{w} = (5, -1, 2, 8)$ be vectors in $\mathbb{R}^4$. Then

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \\ 12 \end{bmatrix}, \quad -3 \cdot \mathbf{v} = \begin{bmatrix} -3 \\ -9 \\ -6 \\ -12 \end{bmatrix}.$$

The next question you might ask is "why do we want to talk about $\mathbb{R}^n$?" $\mathbb{R}^2$ and $\mathbb{R}^3$ have obvious geometric interpretations, but it's hard to imagine the geometry of $\mathbb{R}^4$, and far harder to imagine the geometry of $\mathbb{R}^{300}$, or think of what that might describe. I visit very few three hundred dimensional spaces in my life.

And it's true that when we want to talk about "geometry" per se we will find ourselves returning to $\mathbb{R}^2$ and $\mathbb{R}^3$; throughout the course I will be giving low-dimensional examples so you have pictures to mentally reference, and we will do some work on specifically three-dimensional geometry.

But it turns out that a lot of very interesting things we care about "look like" $\mathbb{R}^n$ in a very specific way. We've already seen one example, in the set of solutions to a system of linear equations. Even if a four-dimensional space doesn't make much sense, a set of equations with four variables certainly does!

But there are many other examples, and in section 2.3 we will talk about what it means to look like $\mathbb{R}^n$ in this way.

2.3 Vector Spaces

We will now define the main object we'll be studying in this course. The following definition will look long and cumbersome; it is our first venture into the *formal* perspective we mentioned on the first day of the course.

The important thing to remember is that we're describing things that look like $\mathbb{R}^n$, or like the set of solutions to a homogeneous system of linear equations; so if you get confused, think about those examples for comparison.

Definition 2.19. Let V be a set together with two operations:

- A *vector addition* which allows you to add two elements of V and get a new element of V. If $\mathbf{v}, \mathbf{w} \in V$ then the sum is denoted $\mathbf{v} + \mathbf{w}$ and must also be an element of V.
- A *scalar multiplication* which allows you to multiply an element of V by a real number (or "scalar") and get a new element of V. If $r \in \mathbb{R}$ and $\mathbf{v} \in V$ then the scalar multiple is denoted $r \cdot \mathbf{v}$ and must also be an element of V.

Further, suppose the following axioms hold for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and any $r, s \in \mathbb{R}$:

1. (Closure under addition) $\mathbf{u} + \mathbf{v} \in V$
2. (Additive commutativity) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. (Additive associativity) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. (Additive identity) There is an element $\mathbf{0} \in V$ called the "zero vector", such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for every $\mathbf{u}$.
5. (Additive inverses) For each $\mathbf{u} \in V$ there is another element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. (Closue under scalar multiplication) $r\mathbf{u} \in V$
7. (Distributivity) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$
8. (Distributivity) $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$
9. (Multiplicative associativity) $r(s\mathbf{u}) = (rs)\mathbf{u}$
10. (Multiplicative Identity) $1\mathbf{u} = \mathbf{u}$.

Then we say V is a *Vector Space*, and we call its elements *vectors*.

Example 2.20. $\mathbb{R}^n$ is a vector space, with the previously defined vector addition and scalar multiplication. We check:

Let $\mathbf{u} = (u_1, \dots, u_n).\mathbf{v} = (v_1, \dots, v_n), \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n, \quad r, s \in \mathbb{R}$. Then, knowing the usual rules of commutativity and associativity of basic arithmetic, we can compute:

1. $\mathbf{u} + \mathbf{v} = (u_1, \dots, u_n) + (v_1, \dots, v_n) = (u_1 + v_1, \dots, u_n + v_n) \in \mathbb{R}^n$.

2.
$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, \dots, u_n) + (v_1, \dots, v_n) = (u_1 + v_1, \dots, u_n + v_n) \\ &= (v_1 + u_1, \dots, v_n + u_n) = (v_1, \dots, v_n) + (u_1, \dots, u_n) = \mathbf{v} + \mathbf{u}\end{aligned}$$

3.
$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= (u_1 + v_1, \dots, u_n + v_n) + (w_1, \dots, w_n) = (v_1 + u_1 + w_1, \dots, v_n + u_n + w_n) \\ &= (v_1, \dots, v_n) + (u_1 + w_1, \dots, u_n + w_n) = \mathbf{v} + (\mathbf{u} + \mathbf{w})\end{aligned}$$

4. We have $\mathbf{0} = (0, \dots, 0)$. Then
$$\mathbf{0} + \mathbf{v} = (0 + v_1, \dots, 0 + v_n) = (v_1, \dots, v_n) = \mathbf{v}.$$

5. Set $-\mathbf{u} = (-u_1, \dots, -u_n)$. Then
$$\mathbf{u} + (-\mathbf{u}) = (u_1 + (-u_1), \dots, u_n + (-u_n)) = (0, \dots, 0) = \mathbf{0}.$$

6.
$$r\mathbf{u} = r(u_1, \dots, u_n) = (ru_1, \dots, ru_n) \in \mathbb{R}.$$

7.
$$\begin{aligned}r(\mathbf{u} + \mathbf{v}) &= r(u_1 + v_1, \dots, u_n + v_n) = (r(u_1 + v_1), \dots, r(u_n + v_n)) \\ &= (ru_1 + rv_1, \dots, ru_n + rv_n) = (ru_1, \dots, ru_n) + (rv_1, \dots, rv_n) = r\mathbf{u} + r\mathbf{v}.\end{aligned}$$

8.
$$\begin{aligned}(r + s)\mathbf{u} &= (r + s)(u_1, \dots, u_n) = ((r + s)u_1, \dots, (r + s)u_n) \\ &= (ru_1 + su_1, \dots, ru_n + su_n) = (ru_1, \dots, ru_n) + (su_1, \dots, su_n) = r\mathbf{u} + s\mathbf{u}.\end{aligned}$$

9.

$$r(s\mathbf{u}) = r(su_1, \dots, su_n) = (rsu_1, \dots, rsu_n) = rs(u_1, \dots, u_n).$$

10.

$$1\mathbf{u} = 1(u_1, \dots, u_n) = (1 \cdot u_1, \dots, 1 \cdot u_n) = (u_1, \dots, u_n) = \mathbf{u}.$$

Remark 2.21. That took forever and was incredibly tedious. (It's not actually *difficult*, just extremely annoying). I will ask you to do this exactly once during this entire course.

So what else is a vector space and "looks like $\mathbb{R}^n$"?

Example 2.22. Let $\mathcal{P}(x) = \{a_0 + a_1x + \cdots + a_nx^n : n \in \mathbb{N}, a_i \in \mathbb{R}\}$ be the set of polynomials with real coefficients. Define addition by

$$(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and define scalar multiplication by

$$r(a_0 + a_1x + \cdots + a_nx^n) = ra_0 + ra_1x + \cdots + ra_nx^n.$$

Then $\mathcal{P}(x)$ is a vector space.

Example 2.23. Let $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \mathcal{F}$ be the set of functions from $\mathbb{R}$ to $\mathbb{R}$—that is, functions that take in a real number and return a real number, the vanilla functions of single-variable calculus. Define addition by $(f + g)(x) = f(x) + g(x)$ and define scalar multiplication by $(rf)(x) = r \cdot f(x)$. Then $\mathcal{F}$ is a vector space.

1. We have vector addition defined by $(f+g)(x) = f(x)+g(x)$. This does give a function, so the vector space is closed under addition.

2. $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.

3. $((f + g) + h)(x) = (f + g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g + h)(x) = (f + (g + h))(x)$.

4. Let $\mathbf{0}$ be the zero function defined by $\mathbf{0}(x) = 0$. Then we see that $(f + \mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x) + 0 = f(x)$.

5. Define $(-f)(x)$ by $(-f)(x) = -f(x)$. Then $(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = \mathbf{0}(x)$.

6. Define scalar multiplication by $(rf)(x) = rf(x)$. This does give a function, so the vector space is closed under multiplication.

7. $(r(f+g))(x) = r(f+g)(x) = r(f(x)+g(x)) = rf(x)+rg(x) = (rf)(x)+(rg)(x)$.

8. $((r+s)f)(x) = (r+s)f(x) = rf(x)+sf(x) = (rf)(x)+(sf)(x)$.

9. $(r(sf))(x) = r(sf)(x) = rsf(x) = (rs)f(x) = ((rs)f)(x)$.

10. $(1 \cdot f)(x) = 1f(x) = f(x)$.

Thus $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space.

Example 2.24. If $A\mathbf{x} = \mathbf{0}$ is a homogeneous system of linear equations, then the set of solutions $N(A)$ is a vector space. I won't prove this now because we will shortly develop techniques to make proving this much faster and less irritating in section 2.5.

Example 2.25. The set $M_{m\times n}$ of $m \times n$ matrices is a vector space under the addition and scalar multiplication defined in section 1.4.1, with zero vector given by

$$\mathbf{0} = (0) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

I'm not going to prove this, but you can see that it should be true for the same reason $\mathbb{R}^{mn}$ is a vector space: they're both just lists of real numbers, but one is arranged in a column and the other in a rectangle. The operations are the same.

Example 2.26. The integers $\mathbb{Z}$ are *not* a vector space (under the usual definitions of addition and multiplication). For instance, $1 \in \mathbb{Z}$ but $.5 \cdot 1 = .5 \not\in \mathbb{Z}$.

(We only need to find one axiom that doesn't hold to show that a set is not a vector space, since a vector space must satisfy all the axioms).

Example 2.27. The closed interval $[0,5]$ is not a vector space (under the usual operations) , since $3, 4 \in [0,5]$ but $3+4 = 7 \not\in [0,5]$.

Example 2.28. Let $V = \mathbb{R}$ with scalar multiplication given by $r \cdot x = rx$ and addition given by $x \oplus y = 2x+y$. Then V is not a vector space, since $x \oplus y = 2x+y \neq 2y+x = y \oplus x$; in particular, we see that $3 \oplus 5 = 11$ but $5 \oplus 3 = 13$.

There are many more examples of vector spaces, but as you can see it's fairly tedious to prove that any particular thing is a vector space. In section 2.5 we'll develop a *much* easier way to establish that something is a vector space, so we won't develop any more examples now.

2.4 Properties of Vector Spaces

The great thing about the formal approach is that we can show that anything that satisfies the axioms of a vector space must aso follow some other rules. We'll establish a few of those rules here, and you will establish a few more in your homework. Of course, there's a sense in which the entire rest of this course will be spent establishing those rules.

Proposition 2.29 (Cancellation)**.** *Let V be a vector space and suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ are vectors. If $\mathbf{u}+\mathbf{w} = \mathbf{v}+\mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.*

Proof. By axiom we know that $\mathbf{w}$ has an additive inverse $-\mathbf{w}$. Then we have

$$\begin{aligned}
\mathbf{u}+\mathbf{w} &= \mathbf{v}+\mathbf{w} \\
(\mathbf{u}+\mathbf{w}) + (-\mathbf{w}) &= (\mathbf{v}+\mathbf{w}) + (-\mathbf{w}) \\
\mathbf{u} + (\mathbf{w} + (-\mathbf{w})) &= \mathbf{v} + (\mathbf{w} + (-\mathbf{w})) && \text{Additive associativity} \\
\mathbf{u} + \mathbf{0} &= \mathbf{v} + \mathbf{0} && \text{Additive inverses} \\
\mathbf{u} &= \mathbf{v} && \text{Additive identity.}
\end{aligned}$$

□

Proposition 2.30. *The additive inverse $-\mathbf{v}$ of a vector $\mathbf{v}$ is unique. That is, if $\mathbf{v}+\mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.*

Proof. Suppose $\mathbf{v}+\mathbf{u} = \mathbf{0}$. By the additive inverses property we know that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, and thus $\mathbf{v}+\mathbf{u} = \mathbf{v} + (-\mathbf{v})$. By cancellation we have $\mathbf{u} = -\mathbf{v}$. □

Remark 2.31. In our axioms we asserted that every vector *has* an inverse, but didn't require that there be only one.

Proposition 2.32. *Suppose V is a vector space with $\mathbf{u} \in V$ a vector and $r \in \mathbb{R}$ a scalar. Then:*

1. $0\mathbf{u} = \mathbf{0}$

2. $r\mathbf{0} = \mathbf{0}$

3. $(-1)\mathbf{u} = -\mathbf{u}$.

Remark 2.33. We would actually be pretty sad if any of those statements were false, since it would make our notation look very strange. (Especially the last statement). The fact that these statements *are* true justifies us using the notation we use.

Proof. 1.

$$\begin{aligned} \mathbf{u} = 1 \cdot \mathbf{u} &= (0+1)\mathbf{u} && \text{Multiplicative identity} \\ &= 0\mathbf{u} + 1\mathbf{u} && \text{Distributivity} \\ &= 0\mathbf{u} + \mathbf{u} && \text{Multiplicative identity} \\ \mathbf{0} + \mathbf{u} &= 0\mathbf{u} + \mathbf{u} && \text{Additive identity} \\ \mathbf{0} &= 0\mathbf{u} && \text{Cancellation} \end{aligned}$$

2. We know that $\mathbf{0} = \mathbf{0}+\mathbf{0}$ by additive identity, so $r\mathbf{0} = r(\mathbf{0}+\mathbf{0}) = r\mathbf{0}+r\mathbf{0}$ by distributivity. Then we have

$$\begin{aligned} \mathbf{0} + r\mathbf{0} &= r\mathbf{0} + r\mathbf{0} && \text{additive identity} \\ \mathbf{0} &= r\mathbf{0} && \text{cancellation.} \end{aligned}$$

3. We have

$$\begin{aligned} \mathbf{v} + (-1)\mathbf{v} &= 1\mathbf{1} + (-1)\mathbf{v} && \text{multiplicative inverses} \\ &= (1 + (-1))\mathbf{v} && \text{distributivity} \\ &= 0\mathbf{v} = \mathbf{0}. \end{aligned}$$

Then by uniqueness of additive inverses, we have $(-1)\mathbf{v} = -\mathbf{v}$.

□

Example 2.34. We'll give one last example of a vector space, which is both important and silly.

We define the *zero vector space* to be the set $\{\mathbf{0}\}$ with addition given by $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and scalar multiplication given by $r \cdot \mathbf{0} = \mathbf{0}$. It's easy to check that this is in fact a vector space.

Notice that we didn't ask what "kind" of object this is; we just said it has the zero vector and nothing else. As such, this could be the zero vector of any vector space at all. In section 2.5 we will talk about vector spaces that fit inside other vector spaces, like this one.

2.5 Subspaces

Our very first two examples of a vector space were the Cartesian plane and Euclidean threespace. But we see that while we can think of them as totally distinct vector spaces, the plane sits *inside* threespace, as a subset. In fact it sits inside it in a number of different ways; we can start by taking the xy plane, the xz plane, or the yz plane.

Similarly, though I haven't proven it yet, we also said that the set of solutions to a homogeneous system of linear equations forms a vector space. But if our system has three variables, then the set of solutions is contained in $\mathbb{R}^3$.

Every vector space has a number of "smaller" vector spaces sitting inside of it. In this section we will study "subspaces", which are vector spaces that are subsets of another vector space. They will be helpful in a number of ways; among these, the easiest way to show that a new object is a vector space is to show that it is a subspace of a vector space we already understand.

Definition 2.35. Let V be a vector space. A subset $W \subseteq V$ is a *subspace* of V if W is also a vector space with the same operations as V.

Example 2.36. The Cartesian plane $\mathbb{R}^2$ is a subset of threespace $\mathbb{R}^3$. Similarly the line $\mathbb{R}^1$ is a subset of the plane $\mathbb{R}^2$. (And we can stack this up as high as we want; $\mathbb{R}^7 \subseteq \mathbb{R}^8$.

Example 2.37. Let $V = \mathbb{R}^3$ and let $W = \{(x, y, x + y) \in \mathbb{R}^3\}$. Geometrically, this is a plane (given by $z = x + y$). We could in fact write $W = \{(x, y, z) : z = x + y\}$; this is a more useful way to write it for multivariable calculus, but less useful for lienar algebra. W is certainly a subset of V, so we just need to figure out if W is a subspace.

We could do this by checking all ten axioms, but that would take a very long time; we want a better tool. And it seems like we should be able to avoid a lot of that work since we *already* know many of the axioms hold in $\mathbb{R}^3$.

Proposition 2.38. *Let V be a vector space and $W \subseteq V$. Then W is a subspace of V if and only if the following three "subspace" conditions hold:*

1. $\mathbf{0} \in W$ *(zero vector);*
2. *Whenever* $\mathbf{u}, \mathbf{v} \in W$ *then* $\mathbf{u} + \mathbf{v} \in W$ *(Closed under addition); and*
3. *Whenever* $r \in \mathbb{R}$ *and* $\mathbf{u} \in W$ *then* $r\mathbf{u} \in W$ *(Closed under scalar multiplication).*

Proof. Suppose W is a subspace of V. Then W is a vector space, so it contains a zero vector and is closed under addition and multiplication by the definition of vector spaces.

Conversely, suppose $W \subseteq V$ and the three subspace conditions hold. We need to check the ten axioms of a vector space. But most of these properties are inherited from the fact that any element of W is also an element of V, and W has the same operations as V.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W$ (and thus $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$), and $r, s \in \mathbb{R}$.

1. W is closed under addition by hypothesis.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ since V is a vector space.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ since V is a vector space.
4. $\mathbf{0} \in W$ by hypothesis, and $\mathbf{u} + \mathbf{0} = \mathbf{u}$ since V is a vector space.
5. $-\mathbf{u} = (-1)\mathbf{u} \in W$ by closure under scalar multiplication.
6. W is closed under scalar multiplication by hypothesis.
7. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ since V is a vector space.
8. $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$ since V is a vector space.
9. $(rs)\mathbf{u} = r(s\mathbf{u})$ since V is a vector space.
10. $1\mathbf{u} = \mathbf{u}$ since V is a vector space.

Thus W satisfies the axioms of a vector space, and is itself a vector space. □

Example 2.39. To continue our earlier example of $W = \{(x, y, x + y)\}$, we only need to check three things. If $(x_1, y_1, x_1 + y_1), (x_2, y_2, x_2 + y_2) \in W$ then

$$\begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) + (y_1 + y_2) \end{bmatrix} \in W.$$

If $r \in \mathbb{R}$, then

$$r \begin{bmatrix} x \\ y \\ x + y \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ (rx) + (ry) \end{bmatrix} \in W.$$

And the zero vector is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0+0 \end{bmatrix} \in W.$$

Thus W is a subspace of V.

Example 2.40. If V is a vector space, then 0 and V are both subspaces of V. We don't actually need to check anything here, since both are clearly subsets of V, and both are already known to be vector spaces.

(When we want to ignore this possibility we will refer to "proper" subspaces, which are neither the trivial space nor the entire space).

Example 2.41. Let $V = \mathbb{R}^2$ and let $W = \{(x, x^2)\} = \{(x, y) : y = x^2\} \subseteq V$. Then W is *not* a subspace (and thus not a vector space):

W does in fact contain the zero vector $(0, 0) = (0, 0^2)$. But we see that $(1, 1) \in W$, and $(1, 1) + (1, 1) = (2, 2) \notin W$. Thus W is not a subspace.

Example 2.42. Let $V = \mathbb{R}^3$ and let $W = \{(x, 0, x) \in \mathbb{R}^3\}$. Is W a subspace of $\mathbb{R}^3$?

We need to check three things.

1. $(0, 0, 0) \in W$ ("by inspection", which basically means "look at it and see that this is true").
2. If $(x, 0, x), (y, 0, y) \in W$, then $(x, 0, x) + (y, 0, y) = (x + y, 0, x + y) \in W$.
3. If $r \in \mathbb{R}$ and $(x, 0, x) \in W$ then $r(x, 0, x) = (rx, 0, rx) \in W$.

Example 2.43. Now let $V = \mathbb{R}^3$ and let $W = \{(x, 1, x) \in \mathbb{R}^3\}$. Is W a subspace of $\mathbb{R}^3$?

We need to check the three properties. But we see in fact that $(0, 0, 0) \notin W$ so this is not a subspace.

Corollary 2.44. *If $A\mathbf{x} = \mathbf{0}$ is a homogeneous system of linear equations, and U is the set of solutions to this system, then U is a subspace of $\mathbb{R}^n$.*

Proof. This follows from proposition 1.49. □

Remark 2.45. The converse is also true: every subspace of $\mathbb{R}^n$ is the set of solutions to some homogeneous system of linear equations. We won't prove this until later.

Example 2.46. Let's look at our earlier examples again. We took $W \subset \mathbb{R}^3$ defined by $W = \{x, y, x+y\}$. This is precisely the set of solutions to the linear equation $x + y - z = 0$.

We also had the subspace given by $W = \{x, 0, x)\}$. This is the solution to the system of linear equations

$$\begin{aligned} x - z &= 0 \\ y &= 0. \end{aligned}$$

In contrast, we can see that $\{(x, 1, x)\}$ is the solution to the system

$$\begin{aligned} x - z &= 0 \\ y &= 1 \end{aligned}$$

which is not homogeneous. And $W = \{(x, x^2)\}$ is the solution to the equation $x^2 - y = 0$, which *is* homogeneous, but isn't *linear*. We saw that neither of these is a vector space.

Example 2.47. Let $V = \mathcal{P}(x)$ and let $W = \{a_1 x + \cdots + a_n x^n\} = x\mathcal{P}(x)$ be the set of polynomials with zero constant term. Is W a subspace of V?

1. The zero polynomial $0 + 0x + \cdots + 0x^n = 0$ certainly has zero constant term, so is in W.

2. If $a_1 x + \cdots + a_n x^n$ and $b_1 x + \cdots + b_n x^n \in W$, then

$$(a_1 x + \cdots + a_n x^n) + (b_1 x + \cdots + b_n x^n) = (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \in W.$$

   Alternatively, we can say that if we add two polynomials with zero constant term, their sum will have zero constant term.

3. If $r \in \mathbb{R}$ and $a_1 x + \cdots + a_n x^n \in W$, then

$$r\,(a_1 x + \cdots + a_n x^n) = (ra_1)x + \cdots + (ra_n)x^n$$

   has zero constant term and is in W.

Thus W is a subspace of V.

Example 2.48. Let $V = \mathcal{P}(x)$ and let $W = \{a_0 + a_1 x\}$ be the space of linear polynomials. Then W is a subspace of V.

1. The zero polynomial $0 + 0x \in W$.

2. If $a_0 + a_1x, b_0 + b_1x \in W$, then $(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x \in W$.

3. If $r \in \mathbb{R}$ and $a_0 + a_1x \in W$, then $r(a_0 + a_1x) = ra_0 + (ra_1)x \in W$.

Example 2.49. Let $V = \mathcal{P}(x)$ and let $W = \{1 + ax\}$ be the space of linear polynomials with constant term 1. Is W a subspace of V?

No, because $0 = 0 + 0x \notin W$.

Exercise 2.50. *Fix a natural number $n \geq 0$. Let $V = \mathcal{P}(x)$ and let $W = \mathcal{P}_n(x) = \{a_0 + a_1x + \cdots + a_nx^n\}$ be the set of polynomials with degree at most n. Then $\mathcal{P}_n(x)$ is a subspace of $\mathcal{P}(x)$.*

Example 2.51. Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of functions of one real variable, and let $W = \mathcal{D}(\mathbb{R}, \mathbb{R})$ be the space of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Is W a subspace of V?

1. The zero function is differentiable, so the zero vector is in W.

2. From calculus we know that the derivative of the sums is the sum of the derivatives; thus the sum of differentiable functions is differentiable. That is, $(f+g)'(x) = f'(x)+g'(x)$.

   So if $f, g \in W$, then f and g are differentiable, and thus $f + g$ is differentiable and thus in W.

3. Again we know that $(rf)'(x) = rf'(x)$. If f is in W, then f is differentiable. Thus rf is differentiable and therefore in W.

Example 2.52. Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and let $W = \mathcal{F}([a, b], \mathbb{R})$ be the space of functions from the closed interval $[a, b]$ to $\mathbb{R}$. We can view W as a subset of V by, say, looking at all the functions that are zero outside of $[a, b]$. Is W a subspace of V?

1. The zero function is in W.

2. If f and g are functions from $[a, b] \to \mathbb{R}$, then $(f + g)$ is as well.

3. If f is a function from $[a, b] \to \mathbb{R}$, then rf is as well.

Example 2.53. Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and let $W = \mathcal{F}(\mathbb{R}, [a, b])$ be the space of functions from $\mathbb{R}$ to the closed interval $[a, b]$. Is W a subspace of V?

No! The simplest condition to check is scalar multiplication. Let $f(x) = b$ be a function in V. Let $r = (b + 1)/b$. Then $(rf)(x) = fb = b + 1$ and thus $rf \notin W$.

3 Spanning sets, linear independence, and bases

We have defined many vector spaces, but we started by looking at $\mathbb{R}^n$, which is much easier to think about. One of the nicest and most helpful things about $\mathbb{R}^n$ is the existence of *coordinates*. Rather than, say, just drawing a point on a graph, or perhaps giving an angle and a distance, we can specify a point in $\mathbb{R}^3$ by giving its x-coordinate, its y-coordinate, and its z-coordinate. And similarly, we can specify a point in $\mathbb{R}^7$ by specifying seven real-number coordinates.

In contrast, it's not really clear what it means to talk about coordinates for $\mathcal{F}(\mathbb{R}, \mathbb{R})$. But if we had coordinates there, it would make our life much easier. (In particular, physicists often want to talk about subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ and then put coordinates on them and treat them like $\mathbb{R}^n$). So we would like to find a way to put coordinates on any vector space V.

We'll see that any "coordinate system" will need to have two basic properties: first, we want it to represent any vector in our vector space; second, we want it to represent each vector only once. We will treat these two criteria separately, and then show that we can always find a set that has both properties, which we will call a "basis".

3.1 Spanning sets

Definition 3.1. If V is a vector space $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a list of vectors in V, then a *linear combination* of of the vectors in S is a vector of the form

$$\sum_{i=1}^{n} a_i \mathbf{v}_i = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where $a_i \in \mathbb{R}$ are (real number) scalars.

A linear combination of vectors in V will always itself be an element of V, since V is closed under scalar multiplication and under vector addition.

Geometrically, a linear combination of vectors represents some destination you can reach only going in the directions of your chosen vectors (for any distance. So if I can go north or west, any distance"northwest" will be a linear combination of those vectors. And "southeast" will as well, since we can always go in the "opposite" direction. But "up" will not be.

Remark 3.2. This is a "linear" combination because it combines the vectors in the same way a line or plane does—adding all the vectors together, but with some coefficient. We will revisit this terminology in the next section when we discuss linear functions.

It's totally possible to have a linear combination of infinitely many vectors. But studying these requires some sense of convergence, and thus calculus/analysis. So we won't talk about it in *this* class, except for the occasional aside.

Example 3.3. Let $V = \mathbb{R}^3$ and let $S = \{(1, 0, 0), (0, 1, 0)\}$. Then we see that

$$(3, 2, 0) = 3(1, 0, 0) + 2(0, 1, 0) \quad \text{and} \quad (-5, 3\pi, 0) = -5(1, 0, 0) + 3\pi(0, 1, 0)$$

are linear combinations of vectors in S.

However, $(1, 1, 1)$ is *not* a linear combination of vectors in S. If it were, we would have

$$a(1, 0, 0) + b(0, 1, 0) = (1, 1, 1)$$

and thus $(a, b, 0) = (1, 1, 1)$ which cannot happen for any $a, b \in \mathbb{R}$.

We see that this idea can tell us how coordinates work: a set of coordinates for V is a set of vectors S where we can build any vector in V as a linear combination of vectors in S. So the next natural question is to take a set S and ask what vectors we can get by taking linear combinations of vectors in S.

Definition 3.4. Let V be a vector space $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of vectors in V. We say the *span* of S is the set of all linear combinations of vectors in S, and write it $\operatorname{span}(S)$ or $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

For notational consistency, we define the span of the empty set $\operatorname{span}(\{\})$ to be the trivial vector space $\mathbf{0} = \{\mathbf{0}\}$.

Example 3.5. As before, take $V = \mathbb{R}^3$ and $S = \{(1, 0, 0), (0, 1, 0)\}$. Then

$$\operatorname{span}(S) = \{a(1, 0, 0) + b(0, 1, 0)\} = \{(a, b, 0)\}.$$

Now let $T = \{(3, 2, 0), (13, 7, 0)\}$. Then

$$\operatorname{span}(T) = \{a(3, 2, 0) + b(13, 7, 0)\} = \{(3a + 13b, 2a + 7b, 0)\}.$$

Notice that these are actually the same set! The first spanning set "looks" nicer, but it's hard to make this sense of "nice" mathematically precise. We'll do our best, but won't really get there until section 7.

However, we *can* tell whether two sets of vectors have the same span pretty easily. This requires us to define a new term:

Definition 3.6. If $A = (a_{ij})$ is a $m \times n$ matrix, then each row can be viewed as a vector in $\mathbb{R}^n$; we call these vectors the *row vectors* of A. We may notate them as $\mathbf{r}_i = (a_{i1}, a_{i2}, \dots, a_{in})$. The span of the row vectors of A is the *row space* of A.

Now we can use what we learned about matrices in section 1.3 to figure out the span of a set of vectors.

Proposition 3.7. *Two row-equivalent matrices have the same row space.*

Proof. We need to check that each elementary row operation doesn't change the span of the set of vectors.

I. (Switch two rows) Switching the order of two vectors does not affect the span at all.

II. (Multiply a row by a nonzero scalar) Multiplying a vector by a non-zero scalar won't change the span of the set of vectors, since in any linear combination we can always just multiply the relevant coefficient by the inverse of our non-zero scalar.

III. (Add a multiple of one row to another) This won't add anything to the span, since a linear combination of the new vectors will still be a linear combination of the old vectors.

This won't lose anything from the span, since we can undo the row operation, and so every old vector is a linear combination of new vectors.

□

Example 3.8. $\operatorname{span}(T) = \operatorname{span}(\{(3,2,0),(13,7,0)\})$ is the rowspace of

$$\begin{bmatrix} 3 & 2 & 0 \\ 13 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 0 \\ 39 & 21 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 0 \\ 0 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus we see that $\operatorname{span}(\{(3,2,0),(13,7,0)\}) = \operatorname{span}(\{(1,0,0),(0,1,0)\})$.

Example 3.9. Take $V = \mathbb{R}^3$ and let $U = \{(1,0,0),(0,1,0),(1,1,0)\}$. What is $\operatorname{span}(U)$?

We see that

$$\operatorname{span}(U) = \{a(1,0,0) + b(0,1,0) + c(1,1,0)\} = \{(a+c, b+c, 0)\}.$$

In this case, it's not too hard to see that this is the same as the set $\operatorname{span}(\{(1,0,0),(0,1,0)\}) = \{(a,b,0)\}$. But we can also use row reduction:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have now described the set $\{(a, b, 0)\}$ as the span of three different sets. Two of these sets have had two elements, and one has had three. It's not too difficult to describe it as the span of a set that's as large as we want: for instance, if we take $S = \{(a, b, 0)\}$ to be the set of all vectors with third coordinate zero, then a little thought will tell us that $\text{span}(S) = S$. At the other extreme, it turns out we need to start with at least two elements to span all of S; we will prove this in section 3.2.

We've discussed the idea of spanning algebraically; what is happening geometrically? Recall that each vector gives us a direction and a distance. Since we can multiply our vectors by any scalar, that means we can go any distance in that direction. And since we can add vectors together, that means we can go in one direction, and then another direction. So the span of a set is all of the places I can get to by only going in the direction of vectors in that set.

Let's return to our first example. Our set S included the vectors $(1, 0, 0)$ and $(0, 1, 0)$; those correspond to "north" and "east" in Euclidean threespace. So the span of S is the set of all locations I can get to by going some distance north and then some distance east. But neither of these vectors moves me at all up and down, so I cannot change my height.

In our third example, we had the vectors $(1, 0, 0), (0, 1, 0), (1, 1, 0)$. So we can go north, or east, or north-east. But north-east doesn't open any new options since we could already go north and then east. And we still can't change our height.

Example 3.10. Let $S = \{(1, 2, 3, 4), (1, 1, 1, 1)\}$. Is $(0, 0, 2, 2)$ in $\text{span}(S)$? Is $(0, 1, 2, 3)$?

Each question like this is really asking us to solve a system of linear equations, and thus we can easily solve it using row reduction.

If $(0, 0, 2, 2) \in \text{span}(S)$ then we can write $(0, 0, 2, 2) = a(1, 2, 3, 4) + b(1, 1, 1, 1)$. Then we have the system

$$\begin{aligned} a + b &= 0 \\ 2a + b &= 0 \\ 3a + b &= 2 \\ 4a + b &= 2 \end{aligned}$$

which gives us the matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 2 \\ 4 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \\ 3 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{array}\right].$$

This system isn't solvable, since the last two equations are $0 = 2$. Thus $(0, 0, 2, 2) \notin \operatorname{span}(S)$.

In contrast, we can easily show that $(0, 1, 2, 3) \in \operatorname{span}(S)$. If we see a way to write $(0, 1, 2, 3)$ as a linear combination of elements of S, we can just write that down and be done with it. But we can also be systematic. We want to solve the equation $(0, 1, 2, 3) = a(1, 2, 3, 4) + b(1, 1, 1, 1)$, which gives the system

$$\begin{aligned} a + b &= 0 \\ 2a + b &= 1 \\ 3a + b &= 2 \\ 4a + b &= 3 \end{aligned}$$

and the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus our system has the solution $a = 1, b = -1$, and so $(0, 1, 2, 3) \in \operatorname{span}(S)$. We can check that indeed $1 \cdot (1, 2, 3, 4) + (-1)(1, 1, 1, 1) = (0, 1, 2, 3)$.

Example 3.11. Let $S = \{\sin^2, \cos^2, \tan^2\}$. Is $1 \in \operatorname{span}(S)$? Is $\sec^2 \in \operatorname{span}(S)$?

We know from trigonometry that $\sin^2 + \cos^2 = 1$, so $1 \in \operatorname{span}(S)$. Then we know that $\sin^2 + \cos^2 + \tan^2 = 1 + \tan^2 = \sec^2 \in \operatorname{span}(S)$.

Spans are really convenient to work with because the span of any set will always be a subspace.

Proposition 3.12. *If V is a vector space and $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \subset V$, then* $\operatorname{span}(S)$ *is a subspace of V.*

Proof. We know that $S \subset V$, and since any linear combination of vectors in V is itself a vector in V, we know that $\operatorname{span}(S) \subset V$. So we just need to check the three subspace conditions.

1. We know that $0\mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in V$. So we have

$$0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_n = \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{0} \in \operatorname{span}(S)$.

2. Suppose $\mathbf{v}, \mathbf{w} \in \operatorname{span}(S)$. This implies that we can write

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n \qquad \mathbf{w} = b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n$$

for some $a_i, b_i \in \mathbb{R}$. Thus

$$\begin{aligned}\mathbf{v} + \mathbf{w} &= (a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n) + (b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n)\\ &= (a_1 + b_1)\mathbf{u}_1 + \cdots + (a_n + b_n)\mathbf{u}_n \in \operatorname{span}(S).\end{aligned}$$

3. Suppose $r \in \mathbb{R}$ and $\mathbf{v} \in \operatorname{span}(S)$. Then we can write

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n$$

for some $a_i \in \mathbb{R}$. Then

$$r\mathbf{v} = r\left(a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n\right) = (ra_1)\mathbf{u}_1 + \cdots + (ra_n)\mathbf{u}_n \in \operatorname{span}(S).$$

Thus we see that $\operatorname{span}(S)$ is a subspace of V. □

Corollary 3.13. *If A is a $m \times n$ matrix, then the rowspace of A is a subspace of $\mathbb{R}^n$.*

As a result of proposition 3.12, we see that every set spans *some* vector space. In fact, this gives us another way to think of the span of a set.

Corollary 3.14. *If V is a vector space and $S \subseteq V$, then $\operatorname{span}(S)$ is the smallest subspace of V containing S.*

Proof. We just showed in proposition 3.12 that $\operatorname{span}(S)$ is a subspace of V, and of course it contains S. So we just need to show that there's no smaller subspace. In particular, I'll prove that if W is a subspace of V, and $S \subseteq W$, then $\operatorname{span}(S) \subseteq W$.

So suppose W is a subspace of V and $S \subseteq W$. Let $\mathbf{v} \in \operatorname{span}(S)$. The $\mathbf{v}$ is a linear combination of vectors in S. But $S \subseteq W$, so $\mathbf{v}$ is a linear combination of vectors in W, and thus an element of W since W is a vector space. Thus any element of $\operatorname{span}(S)$ is an element of W, so $\operatorname{span}(S) \subseteq W$. □

This idea of spanning allows us to generate a set of "coordinates" for a vector space.

Definition 3.15. The set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subseteq V$ is a *spanning set* for V if every vector in V can be written as a linear combination of vectors in S. That is, S is a spanning set for V if $\operatorname{span}(S) = V$.

Example 3.16. Which of the following are spanning sets for $\mathbb{R}^3$?

1. $\{(1,0,0),(0,1,0),(0,0,1),(2,1,1)\}$
2. $\{(1,0,0),(0,1,0),(2,1,1)\}$
3. $\{(1,1,1),(1,1,0),(0,1,1)\}$
4. $\{(1,1,1),(1,1,0),(0,0,1)\}$
5. $\{(1,2,3),(3,2,1)\}$

We can check this by seeing which vectors we can make as linear combinations of the vectors in each set. Thus for each set, we want to see if we can find coefficients to make (a,b,c) a linear combination of the given vectors.

1.
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_4 \\ \alpha_2 + \alpha_4 \\ \alpha_3 + \alpha_4 \end{bmatrix}$$

thus we need to solve the system of equations

$$a = \alpha_1 + 2\alpha_4 \qquad b = \alpha_2 + \alpha_4 \qquad c = \alpha_3 + \alpha_4$$

We can encode this system in an augmented matrix, and get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & a \\ 0 & 2 & 0 & 1 & b \\ 0 & 0 & 1 & 1 & c \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & a \\ 0 & 1 & 0 & 1/2 & b/2 \\ 0 & 0 & 1 & 1 & c \end{array}\right].$$

This has a free variable, so our solution set is

$$\{(a - 2\alpha_4, b/2 - \alpha_4/2, c - \alpha_4, \alpha_4 : \alpha_4 \in \mathbb{R}\}.$$

Thus our system has a solution for any $a, b, c \in \mathbb{R}$, so this is a spanning set.

You might notice a couple important things from this. First the *columns* of the matrix are the vectors in our set S. This is obvious when you think about it, and makes it easy to write the matrix down. But it also has deep ties to an important result we will cover in section 4.2.

Second, we don't care what the solution is, just whether one exists. If we can get the matrix into row-echelon form and have no self-contradictory rows, then there is *a* solution, and our set spans. So we can basically ignore the right-hand column, since we don't care what the solution actually is.

2. Here we need to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_3 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{bmatrix}$$

thus we need to solve the system of equations

$$a = \alpha_1 + 2\alpha_3 \qquad b = \alpha_2 + \alpha_3 \qquad c = \alpha_3.$$

This gives us the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-2c \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \end{array}\right]$$

and thus we have the solution $\alpha_1 = a - 2c$, $\alpha_2 = b - c$, and $\alpha_3 = c$. A solution exists, so our set is a spanning set.

Again we notice that the only important thing is that none of our rows consist entirely of zeroes.

3. We can use the same approach. We get the (unaugmented) matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Regardless of the right-hand column, this will have a solution, so once again we have a spanning set.

4. This is very similar to the last problem. We try to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_3 \end{bmatrix}$$

which gives us the three equations

$$a = \alpha_1 + \alpha_2 \qquad b = \alpha_1 + \alpha_2 \qquad c = \alpha_1 + \alpha_3.$$

We immediately see that $a = \alpha_1 + \alpha_2 = b$, so we can't get any vectors with $a \neq b$, so this is not a spanning set.

If we set this up as a matrix, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has a row that is all zeroes, so the associated system doesn't have a solution for every (a, b, c).

So what is the span of this set? We saw that $a = b$ for any coefficients we choose, but we can see that we can choose a to be anything we like, and c to be anything we like (e.g. set $\alpha_1 = 0, \alpha_2 = a, \alpha_3 = c$). Thus

$$\operatorname{span}(S) = \{(a, b, c) : a = b\}$$

which is a plane.

If we wanted to be systematic here, we could write our vectors as the *rows* of a matrix, and row-reduce it. We can look at

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the span of these vectors is $\operatorname{span}(\{(1, 1, 0), (0, 0, 1)\}) = \{(a, a, c) : a, c \in \mathbb{R}\}$.

5. We have

$$\left[\begin{array}{cc|c} 3 & 1 & a \\ 2 & 2 & b \\ 1 & 3 & c \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & a-b \\ 2 & 2 & b \\ 1 & 3 & c \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & a-b \\ 0 & 4 & 3b-2a \\ 0 & 4 & c-a+b \end{array}\right]$$
$$\to \left[\begin{array}{cc|c} 1 & -1 & a-b \\ 0 & 1 & 3b/4 - a/2 \\ 0 & 1 & c/4 - a/4 + b/4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & a/2 - b/4 \\ 0 & 1 & 3b/4 - a/2 \\ 0 & 0 & c/4 + a/4 - b/2 \end{array}\right].$$

The third row of the unaugmented matrix is all zeroes, so we can't solve this for all possible coefficients. In particular, when we look at the augmented matrix, we see that the third row tells us $0 = c/4 + a/4 - b/2$; thus the equation has solutions if and only if $b/2 = a/4 + c/4$, or $b = (a + c)/2$.

Note that the failure of this set to span is not surprising, since $\mathbb{R}^3$ is "three-dimensional" and we only started with two possible directions to go.

Remark 3.17. Notice that we sometimes want to use our vectors as the rows of a matrix, and sometimes we want to use them as the columns of a matrix. This is a very important duality, and we'll return to it in more detail in section 4.2.

Example 3.18. Which of the following are spanning sets for $\mathcal{P}_3(x)$?

1. $\{1, x, x^2, x^3\}$

   We need to see if we can write an arbitrary polynomial as a linear combination of these elements. So we write

   $$a_0 + a_1x + a_2x^2 + a_3x^3 = \alpha_0 \cdot 1 + \alpha_1 \cdot x + \alpha_2 \cdot x^2 + \alpha_3 \cdot x^3$$

   which gives us the linear equations

   $$a_0 = \alpha_0 \qquad a_1 = \alpha_1 \qquad a_2 = \alpha_2 \qquad a_3 = \alpha_3$$

   which...come presolved. So there is a solution to this system, and $\text{span}(\{1, x, x^2, x^3\}) = \mathcal{P}_3(x)$.

2. $\{1 + x, x^2 + x^3, x + x^2, 1 + x^3\}$

   As before, we try to write a generic polynomial as a linear combination here. We write

   $$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 &= \alpha_0(1 + x) + \alpha_1(x^2 + x^3) + \alpha_2(x + x^2) + \alpha_3(1 + x^3) \\ &= (\alpha_0 + \alpha_3) + (\alpha_0 + \alpha_2)x + (\alpha_1 + \alpha_2)x^2 + (\alpha_1 + \alpha_3)x^3 \end{aligned}$$

   which gives us the system of equations

   $$a_0 = \alpha_0 + \alpha_3 \qquad a_1 = \alpha_0 + \alpha_2 \qquad a_2 = \alpha_1 + \alpha_2 \qquad a_3 = \alpha_1 + \alpha_3.$$

We can turn this system of equations into a matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a_0 \\ 1 & 0 & 1 & 0 & a_1 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 1 & 0 & 1 & a_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a_0 \\ 0 & 0 & 1 & -1 & a_1 - a_0 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & -1 & 1 & a_3 - a_2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a_0 \\ 0 & 1 & 1 & 0 & a_2 \\ 0 & 0 & 1 & -1 & a_1 - a_0 \\ 0 & 0 & 0 & 0 & a_3 - a_2 + a_1 - a_0 \end{array}\right]$$

We get a row of all zeroes, meaning that the set doesn't span. In particular, we get the constraint that $a_3 - a_2 + a_2 - a_0 = 0$; thus the span is the set of all polynomials where the sum of the even-degree coefficients is the same as the sum of the odd-degree coefficients. And we can go back and check that this is a property that all of our original vectors have, and that is stable under addition and scalar multiplication.

Obviously answering this question effectively requires a thorough study of solving systems of equations like this; we will return to this question in great detail in the future.

We finish with a few facts about spans and spanning sets:

Proposition 3.19. *Let V be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Then*

1. $\mathbf{0} \in \operatorname{span}(S)$.
2. *If $S \subseteq T$ then* $\operatorname{span}(S) \subseteq \operatorname{span}(T)$.
3. *If $\mathbf{u} \in \operatorname{span}(S)$, then we write $S \cup \{\mathbf{u}\} = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}\}$ for the set containing everything in S, and also the element $\mathbf{u}$. Then*

$$\operatorname{span}(S) = \operatorname{span}(S \cup \{\mathbf{u}\}).$$

4. *If W is a subspace of V then* $\operatorname{span}(W) = W$.

Proof. 1. $0 = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n \in \operatorname{span}(S)$.

2. Set $T = \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1, \dots, \mathbf{u}_m\}$. Suppose $\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \in \operatorname{span}(S)$. Then

$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0\mathbf{u}_1 + \dots 0\mathbf{u}_m \in \operatorname{span}(T).$$

3. To prove two sets are equal, it's generally easiest to prove each one is a subset of the other—that is, for each set, we take an arbitrary element of that set and prove it is also an element of the other set.

   We know that $S \subseteq S \cup \{\mathbf{u}\}$, so by part (2) we know that $\operatorname{span}(S) \subseteq \operatorname{span}(S \cup \{\mathbf{u}\})$.

   So suppose $\mathbf{w} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n + c\mathbf{u} \in \operatorname{span}(S)$. We know $\mathbf{u} \in \operatorname{span}(S)$ so we can write $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. Then

$$\begin{aligned}
\mathbf{w} &= b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n + c\mathbf{u} \\
&= b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n + c\,(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) \\
&= (b_1 + ca_1)\mathbf{v}_1 + \cdots + (b_n + ca_n)\mathbf{u}_n \in \operatorname{span}(S).
\end{aligned}$$

   Thus $\operatorname{span}(S) \subseteq \operatorname{span}(S \cup \{\mathbf{u}\})$ and $\operatorname{span}(S \cup \{\mathbf{u}\}) \subseteq \operatorname{span}(S)$, so we know that $\operatorname{span}(S) = \operatorname{span}(S \cup \{\mathbf{u}\})$.

4. We know that $W \subseteq \operatorname{span}(W)$, so we just need to show that $\operatorname{span}(W) \subseteq W$. Let $\mathbf{w} = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n \in \operatorname{span}(W)$, where $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$. Then since W is a vector space, it is closed under linear combinations, so any linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_n$ is in W. Thus in particular $\mathbf{w} \in W$.

□

Corollary 3.20. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set for a vector space V, and $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a set with $S \subseteq \operatorname{span}(T)$, then T is also a spanning set for V.*

Proof. We know that $S \subseteq \operatorname{span}(T)$, and this means that $\operatorname{span}(S) \subseteq \operatorname{span}(\operatorname{span}(T))$. But $\operatorname{span}(\operatorname{span}(T)) = \operatorname{span}(T)$, so we have that $V = \operatorname{span}(S) \subseteq \operatorname{span}(T)$. Thus T spans V.

□

3.2 Linear Independence

This idea of spanning sets answers half of our original question. If we have a spanning set for V, we can write our vectors as sums of elements of the spanning set. But recall we also want this representation to be *unique*—we want to know that if we give two different sets of "coordinates" that they actually represent distinct vectors.

In the previous section, we wanted to study the span of a set of vectors—which tells you how many places you can get with them. Now we want to measure the redundancy: Do we have more vectors in our spanning set than we need?

Definition 3.21. We say a set of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq V$ is *linearly independent* if the only scalars solving the equation

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

are $a_1 = \cdots = a_n = 0$.

If a set of vectors is not linearly independent, we call it *linearly dependent* and there is a *linear dependence* relationship among the vectors.

Example 3.22. 1. The set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent: suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then we have the system of equations $a = 0, b = 0, c = 0$ and thus all the scalars are zero.

2. The set $S = \{(1,0,0),(0,1,0)\}$ is linearly independent. Suppose

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}.$$

Then we have the system of equations $a = 0, b = 0$ and thus all the scalars are zero.

3. The set S $= \{(1,0,0),(0,1,0),(1,1,0)\}$ is not linearly independent, since

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

4. Any set containing the zero vector is linearly dependent, since $1 \cdot \mathbf{0} = \mathbf{0}$ but $1 \neq 0$.

As before, we see that each problem is really asking for the solution to a system of equations. But unlike in section 3.1 we don't need to find a solution for any possible constants. Instead, we just want to see if there's more than one solution to the equation $A\mathbf{x} = \mathbf{0}$.

Example 3.23. Let $S = \{(3,5,1),(3,5,3),(1,1,1)\}$. Then we want to know if there exist a, b, c such that

$$a \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The associated matrix is $A = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ and we want to solve the homogeneous system, so we just have to reduce the matrix: we get

$$\begin{bmatrix} 3 & 3 & 1 \\ 5 & 5 & 1 \\ 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & -6 & -2 \\ 0 & -10 & -4 \\ 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And thus we have the unique solution $a = 0, b = 0, c = 0$. Thus S is linearly independent.

There are a few ways we can think about linear independence. One is that a linearly independent set is one where the zero vector can be expressed uniquely—$\mathbf{0}$ is in the span of any set, but it is only in the span of a linearly independent set in one way. In fact, this is enough to make *every* vector expressed uniquely.

Proposition 3.24. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a subset of a vector space V. Then S is linearly independent if and only if every vector in* span(S) *can be expressed uniquely as a linear combination of vectors in S.*

Proof. Suppose S is linearly dependent. Then by definition of linear independence there are $a_i \in \mathbb{R}$ such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$$

and thus the expression of $\mathbf{0}$ as a linear combination of vectors in S is not unique.

Now suppose not every vector in span(S) can be expressed uniquely as a linear combination of vectors in S. By definition of span, every vector in span(S) can be represented as a linear combination of vectors in S, so it must be the case that some vector is not represented uniquely, and thus can be written as a linear combination of elements of S in two different ways.

Suppose $\mathbf{u}$ is such an element. Then we have

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{u} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$
$$(a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Thus we can write $\mathbf{0}$ as a nontrivial linear combination of elements of S, so S is linearly dependent. $\square$

Another way to think of this is that in a linearly dependent set, we can express one vector as a linear combination of the others, and thus at least one vector in the set is redundant.

This gives us a geometric interpretation as well. Generally, any one vector defines a line containing it and the origin. Two vectors in general define a plane, three vectors a threespace, and so on. A set is linearly independent if the linear space it defines is as big as you would expect. A set is linearly dependent if the set is smaller—if, say, you have points but they're all on the same line through the origin, so you don't actually get a whole plane.

Lemma 3.25. *A set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if some element can be written as a linear combination of the others.*

Proof. Without loss of generality, assume $\mathbf{v}_1$ can be written as a linear combination of $\mathbf{v}_2, \dots, \mathbf{v}_n$. (That is, we're assuming one of the vectors can be written as a linear combination of the others, and since order doesn't matter we can assume that it's $\mathbf{v}_1$ to keep the notation simple). Then we have

$$\begin{aligned}
\mathbf{v}_1 &= a_2\mathbf{v}_2 + \dots a_n\mathbf{v}_n \\
\mathbf{v}_1 - \mathbf{v}_1 &= a_2\mathbf{v}_2 + \dots a_n\mathbf{v}_n - \mathbf{v}_1 \\
\mathbf{0} &= (-1)\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.
\end{aligned}$$

Then we have written $\mathbf{0}$ as a nontrivial linear combination of elements of S, and thus S is linearly dependent.

Conversely, suppose S is linearly dependent. Then there are a_i not all zero such that

$$\mathbf{0} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

We know not all the a_i are zero, so assume without loss of generality that $a_1 \neq 0$. Then we have

$$\begin{aligned}
-a_1\mathbf{v}_1 &= a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \\
\mathbf{v}_1 &= \frac{-a_2}{a_1}\mathbf{v}_2 + \cdots + \frac{-a_n}{a_1}\mathbf{v}_n
\end{aligned}$$

and thus we can write $\mathbf{v}_1$ as a linear combination of the other vectors in S. $\square$

Corollary 3.26. *$S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly dependent if and only if there is some $\mathbf{v}_i$ such that* $\operatorname{span}(S) = \operatorname{span}(\{\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$.

In practice this is how we prefer to test for linear independence: we try to write one vector as a linear combination of the others. Sometimes this is easy and we're done. Other times this is difficult, or we become convinced it's not possible, and then we have to go back to solving linear equations.

Example 3.27. 1. Let $S = \{(1,1,1),(1,1,0),(0,0,1)\}$. We see that $(1,1,1) = (1,1,0) + (0,0,1)$ so this set is linearly dependent.

2. Let $S = \{(1,1,1),(1,1,0),(0,1,1)\}$. It might look like this is similar, and we could write $(1,1,1)$ somehow as a combination of the other two. But we see that's not actually possible. In fact we write

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}a+b\\a+b+c\\b+c\end{bmatrix}$$

and this gives us the system

$$0 = a+b \qquad 0 = a+b+c \qquad 0 = b+c$$

with associated matrix

$$A = \begin{bmatrix}1&1&0\\1&1&1\\1&0&1\end{bmatrix} \to \begin{bmatrix}1&1&0\\0&0&1\\1&0&1\end{bmatrix} \to \begin{bmatrix}1&1&0\\0&0&1\\1&0&0\end{bmatrix} \to \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Thus we get the unique solution $a = b = c = 0$ and so the set is linearly independent.

3. Let $S = \{(1,1,1),(1,1,0),(2,3,1),(0,1,1)\}$. To show linear dependence, we might want to show that one vector is a sum of the others. In fact we cannot write $(1,1,1)$ as a linear combination of the other vectors:

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}2\\3\\1\end{bmatrix} + c\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}a+2b\\a+3b+c\\b+c\end{bmatrix}$$

gives the system

$$\left[\begin{array}{ccc|c}1&2&0&1\\1&3&1&1\\0&1&1&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&0&1\\0&1&1&0\\0&1&1&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&0&1\\0&1&1&0\\0&0&0&1\end{array}\right]$$

and the last row is a contradiction $0 = 1$. Thus there is no solution to this system; we cannot write $(1, 1, 1)$ as a linear combination of the other vectors.

But this doesn't mean that the vectors are linearly independent. Corollary 3.26 says that the vectors are independent if and only if *some* vector is a linear combination of the others. There is in fact a vector that is a sum of some of the others: we see that

$$2\begin{bmatrix}1\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

and thus set S is linearly dependent.

If we didn't see this directly, we could set up the matrix associated to all four vectors:

$$\begin{bmatrix}1&1&2&0\\1&1&3&1\\1&0&1&1\end{bmatrix} \to \begin{bmatrix}1&1&2&0\\0&0&1&1\\0&-1&-1&1\end{bmatrix} \to \begin{bmatrix}1&0&1&1\\0&1&1&-1\\0&0&1&1\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&1&0&-2\\0&0&1&1\end{bmatrix}.$$

We have the fourth column as a free variable, so there is more than one solution. Thus the set is not linearly independent. In particular, if we have a linear combination of these vectors that sums to $\mathbf{0}$, it satisfies $a = 0, b = 2d, c = -d$.

Again we see we're looking at a matrix whose columns are the vectors in the set in question. We want to see if any non-trivial linear combination of our vectors is equal to the zero vector, and that leads to solving a homogeneous system of equations. Thus we get the following result:

Proposition 3.28. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{R}^m$. Let A be the matrix whose columns are the vectors $\mathbf{v}_i$. Then S is linearly independent if and only if $N(A) = \{\mathbf{0}\}$.*

Proof. By definition, S is linearly independent if and only if the only solution to $a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n = \mathbf{0}$ is $(0, 0, \dots, 0)$. But this is a homogeneous system of linear equations whose matrix is A, so $N(A)$ is the set of solutions to this system of equations. Thus S is linearly independent if and only if the only element of $N(A)$ is the zero vector. □

We again conclude with a fact about linear independence.

Proposition 3.29. *If $S \subseteq T$ and T is linearly independent, then S is also linearly independent.*

Proof. Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1, \dots, \mathbf{u}_m\} = T$, and T is linearly independent. Now suppose their are scalars a_i such that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

Then we have

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0\mathbf{u}_1 + \cdots + 0\mathbf{u}_m = 0$$

and since T is linearly independent, we have $a_i = 0$ for every a_i. Thus we see that S is linearly independent.

□

3.3 Vector Space Bases

Having now discussed the two properties we want a coordinate system to have, we can define exactly what we mean by a coordinate system.

Definition 3.30. If V is a vector space and S is a spanning set for V that is also linearly independent, we say that S is a *basis* for V.

Example 3.31. The set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis for $\mathbb{R}^3$, as we have seen before. We call this set the *standard basis* for $\mathbb{R}^3$, and we write the three elements $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We can generalize this to $\mathbb{R}^n$. We define the *standard basis vectors* for $\mathbb{R}^n$ by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \dots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and the set of standard basis vectors is the *standard basis*. You can check that the standard basis is in fact a basis.

Example 3.32. Every (non-trivial) vector space has more than one basis. The set $S = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis for $\mathbb{R}^3$:

First we show that it is a spanning set. Let $(a,b,c) \in \mathbb{R}^3$. Then we want to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which gives the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \end{array}\right]$$

which tells us that $\alpha_3 = c, \alpha_2 = b-c, \alpha_1 = a-b$. Thus there is a solution for any $(a,b,c) \in \mathbb{R}^3$, and the set spans.

We also need to prove linear indepencence. So suppose

$$\mathbf{0} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This gives us a system of linear equations corresponding to the homogeneous system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the only solution here is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus S is linear independent, and since it also spans, it is a basis.

Example 3.33. The set $S = \{(1,0,0),(0,1,0)\}$ is not a basis for $\mathbb{R}^3$. It is linearly independent (since it is a subset of the standard basis, which is linear independent), but it is not a spanning set, since $(0,0,1)$ is not in the span of S.

Example 3.34. The set $S = \{(2,3),(3,4),(4,4)\}$ is a spanning set for $\mathbb{R}^2$ but not a basis. To see that it's a spanning set we solve

$$\begin{bmatrix} a \\ b \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \\ 3\alpha_1 + 4\alpha_2 + 4\alpha_3 \end{bmatrix}$$

giving the system of equations

$$a = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \qquad b = 3\alpha_1 + 4\alpha_2 + 4\alpha_3$$

and the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & a \\ 3 & 4 & 4 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & b-a \\ 2 & 3 & 4 & a \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & b-a \\ 0 & 1 & 4 & 3a-2b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 3b-4a \\ 0 & 1 & 4 & 3a-2b \end{array}\right].$$

Thus for any $(a, b) \in \mathbb{R}^2$, at least one solution exists; in fact we can pick α_3 to be any real number and we get a corresponding solution $(3b - 4a + 4\alpha_3, 3a - 2b - 4\alpha_3, \alpha_3)$. Thus the set spans.

But S is not linearly independent. We can see this in a few ways. Most easily we can observe that $(2, 3) + (1/4)(4, 4) = (3, 4)$. If we can't see that on our own, we can do a couple things. We can find the nullspace:

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix}$$

and we see the nullspace $\{(4\alpha, -4\alpha, \alpha)\}$ is non-trivial, so the set is not linearly independent.

But if these row operations seem familiar, that's because we did exactly the same thing to check spanning! So we can look at our spanning equations and try to find all the solutions when we take $a = b = 0$. We see that there's more than one solution there, so the vectors aren't linearly independent.

Example 3.35. The set $S = \{1, x, x^2, x^3\}$ is a basis for $\mathcal{P}_3(x)$. So is the set $T = \{1 + x + x^2 + x^3, 1 + x + x^2, 1 + x, 1\}$.

Determining whether a set is a basis is sometimes annoying, but doesn't involve anything we haven't already done: a basis is just a set that both spans and is linearly independent, and we can check both properties individually. But we'd like to make things a little simpler.

Further, we want to talk about how "big" a space is, and this should plausibly be determined by how many elements there are in the basis. But since every space has more than one basis, talking about the size of "the" basis is potentially problematic. Fortunately, this is not an actual problem, as we shall see.

Lemma 3.36. *If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans a vector space V, and $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a collection of vectors in V with $m > n$, then T is linearly dependent.*

Proof. There are two possible ways to prove this. One involves simply writing out a bunch of linear equations and solving them; this works, but is more tedious than informative. We'll use a more formal and abstract approach to proving this instead, which, hopefully, will actually explain some of *why* this is true.

We will start with the set S, and one by one we will trade out vectors in S for vectors in T, and show that we always still have a spanning set. We will suppose T is linearly independent, and show that $m \leq n$.

Since S is a spanning set, we know that $\mathbf{u}_1 \in \text{span}(S)$, and thus $\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1\}$ is linearly dependent by lemma 3.25. Then we can rewrite our linear dependence equation to express $\mathbf{v}_1$ (without loss of generality) as a linear combination of $\{\mathbf{u}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = S_1$, and thus

$$\text{span}(S) = \text{span}(\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{u}_1\}) = \text{span}(S_1).$$

We can repeat this process: at every step we add the next vector from T to get the set $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_k, \dots, \mathbf{v}_n\}$. Since S_{k-1} is a spanning set, this set is linearly dependent; since the $\mathbf{u}_i$ are linearly independent by hypothesis, we can remove one of the $\mathbf{v}_i$, and without loss of generality we can remove $\mathbf{v}_k$, to obtain the set $S_k = \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$.

If $m > n$, we can continue until we have replaced every $\mathbf{v}_i$. Then we have $S_n = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a spanning set, and thus $\mathbf{u}_{n+1} \in \text{span}(S_n)$ and so T is linearly dependent, which contradicts our assumption.

Thus if T is linearly independent, we must have $m \leq n$. Conversely, if $m > n$ then T is linearly dependent, as we stated. □

Corollary 3.37. *$S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ are two bases for a space V, then they are the same size, i.e. $m = n$.*

Proof. S is a spanning set and T is linearly independent, so we can't have $m > n$ by lemma 3.36. But T is a spanning set and S is linearly independent, so we can't have $n > m$ by lemma 3.36. Thus $n = m$. □

Definition 3.38. Let V be a vector space. If V has a basis consisting of n vectors, we say that V has *dimension* n and write $\dim V = n$. The trivial vector space $\{\mathbf{0}\}$ has dimension 0.

We say that V is *finite–dimensional* if there is a finite set of vectors that spans V. (Thus if V is n-dimensional it is finite-dimensional). Otherwise, we say that V is *infinite–dimensional*.

In this course we will primarily discuss finite dimensional vector spaces.

Example 3.39. The set of standard basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$, so $\mathbb{R}^n$ is n-dimensional.

The set $\{1, x, \dots, x^n\}$ is a basis for $\mathcal{P}_n(x)$. This set has $n+1$ vectors, so $\dim \mathcal{P}_n(x) = n+1$.

$\mathcal{P}(x)$ does not have a finite basis. We can see this since the set $S = \{1, x, \dots, x^n\}$ is linearly independent for any n; but every spanning set is at least as big as any linearly independent set, so we can never have a finite spanning set. However, if we allow infinite bases, then $\{1, x, \dots, x^n, \dots\}$ is a basis for $\mathcal{P}(x)$.

Remark 3.40. $\mathcal{C}([a,b],\mathbb{R})$ is infinite–dimensional, but if we allow infinite sums and make convergence arguments it is possible to think of the set $\{1, x, \dots, x^n, \dots\}$ as a sort of ("separable") basis. But this requires analysis and is outside the scope of this course. We can also build a (separable) basis out of the functions $\sin(nx)$ and $\cos(nx)$ for $n \in \mathbb{N}$; this is the foundation of Fourier analysis and Fourier series.

The set $\mathcal{F}(\mathbb{R},\mathbb{R})$ is absurdly huge, and does not have a countable basis. If you believe the axiom of choice it has a basis, as all sets do, but you can't possibly write it down. You can think of it has having "coordinates" given by functions like

$$f_r(x) = \begin{cases} 1 & x = r \\ 0 & x \neq r \end{cases}$$

but this isn't a basis because it would require uncountable sums, which you can't really define.

We'd like to make it easier to check if a set is a basis, and easier to find bases for spaces. We show here that if we start with basically any set, we can turn it into a basis.

Lemma 3.41 (Basis Reduction). *Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set for V. Then S can be reduced to a basis for V. That is, there is a subset $B \subseteq S$ that is a basis for V.*

Proof. If S is linearly independent, then it is a basis and we're done.

So suppose S is linearly dependent. Then we know at least one vector is redundant, so without loss of generality we can reorder the set so that we can write $\mathbf{v}_n$ as a linear combination of the other vectors in S.

But then $\operatorname{span}(S) = \operatorname{span}(\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\})$, and $S_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ is a spanning set for V and a proper subset of S. If S_1 is linearly independent, then it is a basis; if not, we can repeat this process until we reach a linearly independent set, which is our basis B. □

Remark 3.42. This proof assumes that S is finite. The result is still (mostly) true if S is infinite, but if the space is finite-dimensional this isn't interesting, and if the space is infinite-dimensional things get very complicated and we don't want to worry about them here.

Example 3.43. Let $S = \{(1,1,0), (1,1,1), (0,0,1), (2,7,0)\}$ be a spanning set for $\mathbb{R}^3$. Find a basis $B \subseteq S$ for $\mathbb{R}^3$.

We'll take as given that this is a spanning set, which is not difficult to check. We see that we can write $(1,1,1) = (1,1,0) + (0,0,1)$, so we can remove $(1,1,1)$ without changing the span, and we have $B = \{(1,1,0), (0,0,1), (2,7,0)\} \subseteq S$ is a basis for $\mathbb{R}^3$.

Example 3.44. Let $S = \{(1,2,3),(1,1,1),(5,-2,1),(-4,3,2)\}$ be a spanning set for $\mathbb{R}^3$. Find a basis $B \subseteq S$ for $\mathbb{R}^3$.

We'll take as given that S is a spanning set. We need to write one vector as a linear combination of the others, which is essentially the same problem as finding a nontrivial linear combination equal to zero. So we set up the equation

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} + c\begin{bmatrix}5\\-2\\1\end{bmatrix} + d\begin{bmatrix}-4\\3\\2\end{bmatrix}$$

which gives us

$$\begin{bmatrix}1&1&5&-4\\2&1&-2&3\\3&1&1&2\end{bmatrix} \to \begin{bmatrix}1&1&5&-4\\0&-1&-12&11\\0&-2&-14&14\end{bmatrix} \to \begin{bmatrix}1&0&-7&7\\0&1&12&-11\\0&0&10&-8\end{bmatrix} \to \begin{bmatrix}1&0&0&7/5\\0&1&0&-7/5\\0&0&1&-4/5\end{bmatrix}$$

This gives us $a = -7/5d, b = 7/5d, c = 4/5d$, or in other words if we set $d = 1$ we get

$$\begin{bmatrix}-4\\3\\2\end{bmatrix} = \frac{7}{5}\begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{7}{5}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{4}{5}\begin{bmatrix}5\\-2\\1\end{bmatrix}.$$

Thus $(-4,3,2)$ can be written as a linear combination of the other vectors, and so we have $B = \{(1,2,3),(1,1,1),(5,-2,1)\}$ is a basis for $\mathbb{R}^3$. We know this is a basis because it is still a spanning set, and has the correct number of elements.

(We could actually have removed any vector from this set and still gotten a basis; each element can be written as a combination of the others, as you can see by rearranging the last equation. But it's sufficient here to find one basis.)

Example 3.45. Let $S = \{1-x, x^2-x, 1+x+x^2, x^2-1\}$ be a spanning set for $\mathbb{P}_2(x)$. Find a basis $B \subset S$.

We need to remove one vector which depends on the others. We need to find a nontrivial linear combination, so we have the equation

$$a(1-x) + b(x^2-x) + c(1+x+x^2) + d(x^2-1) = 0$$

which gives the homogeneous system

$$\begin{bmatrix}1&0&1&-1\\-1&-1&1&0\\0&1&1&1\end{bmatrix} \to \begin{bmatrix}1&0&1&-1\\0&-1&2&-1\\0&1&1&1\end{bmatrix} \to \begin{bmatrix}1&0&1&-1\\0&1&1&1\\0&0&3&0\end{bmatrix} \to \begin{bmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&0\end{bmatrix}$$

which tells us that $a = d, b = -d, c = 0$.

Thus we have $(1-x) - (x^2 - x) + (x^2 - 1) = 0$ and so $x^2 - x = (1-x) + (x^2-1)$, and thus the element $x^2 - x$ is redundant and a linear combination of the other vectors. We can remove it, and get a basis $\{1-x, 1+x+x^2, x^2-1\}$.

Notice here that we could have removed the first element $1-x$ or the fourth element x^2-1, since we can rearrange our equation to write either of those as a linear combination of the others. But we could *not* have removed the element $1+x+x^2$, since we didn't find we could write it as a combination of the others; it was in fact necessary for this set to span.

Lemma 3.46 (Basis Padding). *Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent in V. Then if V has any finite spanning set $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$, we can obtain a basis by padding S. That is, there is a basis B for V with $S \subseteq B$.*

Proof. If $T \subset \operatorname{span}(S)$, then $\operatorname{span}(T) \subset \operatorname{span}(S)$, so S is a spanning set for V and thus a basis, so we're done.

So suppose without loss of generality that $\mathbf{u}_1 \notin \operatorname{span}(S)$. Then $S_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{u}_1\}$ is linearly independent by lemma 3.25 since we can't write any element as a linear combination of the others.

If S_1 spans V, then it is a basis and we're done. If not, there is some other $\mathbf{u}_i \notin \operatorname{span}(S_1)$, so we can repeat the process, and after at most m steps this process will terminate (since we run out of elements in T). When we reach a spanning set, this is our basis.

□

Example 3.47. Let $S = \{(1,1,0), (1,-1,0)\}$. Find a basis $B \supseteq S$ for $\mathbb{R}^3$.

We see that S is linearly independent, so we just need to find a vector that isn't in $\operatorname{span}(S)$. It's clear that $(0,0,1) \notin \operatorname{span}(S)$, so we see that $B = \{(1,1,0), (1,-1,0), (0,0,1)\}$ satisfies our requirements.

But there are many choices we could make. It's also the case that $(1,1,1) \notin \operatorname{span}(S)$, so we see that $B_1 = \{(1,1,0), (1,-1,0), (1,1,1)\}$ also satisfies our requirements.

Example 3.48. Let $T = \{(5,2,-3), (1,-4,7)\}$. Find a basis $B \supseteq T$.

We just need to find a vector that isn't in $\operatorname{span}(T)$. We can make a guess here and prove it by hand; so for instance it looks like $(1,0,0)$ is not in the span. Indeed, we see that if

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} + b \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$$

then we have the system

$$\left[\begin{array}{cc|c} 5 & 1 & 1 \\ 2 & -4 & 0 \\ -3 & 7 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 9 & 1 \\ 0 & -22 & -2 \\ 0 & 34 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2/11 \\ 0 & 1 & 1/11 \\ 0 & 0 & -1/11 \end{array}\right]$$

and thus we have a contradiction.

Thus $(1, 0, 0)$ is not in the span of T, and we have that $B = \{(5, 2, -3), (1, -4, 7), (1, 0, 0)\} \supseteq T$ is a basis for $\mathbb{R}^3$.

Example 3.49. Let $S = \{1 + x, x^2 - 3\} \subset \mathcal{P}_2(x)$. Can we find a basis B for $\mathcal{P}_2(x)$ that contains T?

We need to find a vector (or quadratic polynomial) that isn't in S. There are lots of choices here, but it looks to me like 1 is not in the span of S. Then we check: suppose $a(1 + x) + b(x^2 - 3) = 1$. Then we have

$$(a - 3b) + ax + bx^2 = 1$$

which gives the system

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

which has no solution. Thus indeed $1 \notin \text{span}(S)$, so $\{1, 1 + x, x^2 - 3\}$ is a basis for $\mathcal{P}_3(x)$.

4 Linear Functions

Now that we understand vector spaces a bit more, we want to see how functions between vector spaces work. There are of course lots of functions that that take in vectors and output other vectors; almost any multivariable function technically qualifies. But we actually want to care about functions that in some sense are compatible with the actual vector space structure.

4.1 Definition and examples

Definition 4.1. Let U and V be vector spaces, and let $L : U \to V$ be a function with domain U and codomain V. We say L is a *linear transformation* if:

1. Whenever $\mathbf{u}_1, \mathbf{u}_2 \in U$, then $L(\mathbf{u}_1 + \mathbf{u}_2) = L(\mathbf{u}_1) + L(\mathbf{u}_2)$.
2. Whenever $\mathbf{u} \in U$ and $r \in \mathbb{R}$, then $L(r\mathbf{u}) = rL(\mathbf{u})$.

Example 4.2. If A is a $m \times n$ matrix, then A gives us a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$, given by $A(\mathbf{x}) = A\mathbf{x}$. That is, our input is a (column) vector in $\mathbb{R}^n$, and our output is the vector in $\mathbb{R}^m$ we get by multiplying our column vector by our matrix.

Geometrically, a linear transformation can stretch, rotate, and reflect, but it cannot bend or shift.

Example 4.3. Consider the function from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by a rotation of ninety degrees counterclockwise. We can see by drawing pictures that the sum of two rotated vectors is the rotation of the sum of the vectors, and that the rotation of a streched vector is the same as the strech of a rotated vector. So this is a linear transformation.

Example 4.4. A *translation* is a function defined by $f(\mathbf{x}) = \mathbf{x} + \mathbf{u}$ for some fixed vector $\mathbf{u}$. (Geometrically, it corresponds to sliding or translating your input in the direction and distance of the vector $\mathbf{u}$).

This is *not* a linear transformation. For instance, $f(r\mathbf{x}) = r\mathbf{x} + \mathbf{u} \neq r(\mathbf{x} + \mathbf{u}) = rf(\mathbf{x})$ unless $\mathbf{u} = \mathbf{0}$.

Example 4.5. The function $f(x) = x^2$ is not a linear transformation from $\mathbb{R}$ to $\mathbb{R}$, since $f(2x) = (2x)^2 = 4x^2 \neq 2x^2 = 2f(x)$.

Example 4.6. Define a function $L : \mathbb{R}^3 \to \mathbb{R}^2$ by $L(x, y, z) = (x + y, 2z - x)$. We check:

$$\begin{aligned} L((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= L(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2 + y_1 + y_2, 2z_1 + 2z_2 - x_1 - x_2) \\ &= (x_1 + y_1, 2z_1 - x_1) + (x_2 + y_2, 2z_2 - x_2) \\ &= L(x_1, y_1, z_1) + L(x_2, y_2, z_2). \\ L(r(x, y, z)) &= L(rx, ry, rz) = (rx + ry, 2rz - rx) = \\ &= r(x + y, 2z - x) = rL(x, y, z). \end{aligned}$$

Thus L is a linear transformation by definition.

Definition 4.7. Let $L : U \to V$ be a linear transformation. If $\mathbf{u} \in U$ is a vector, we say the element $L(\mathbf{u}) \in V$ is the *image* of $\mathbf{u}$.

If $S \subset U$ then we define the image of S to be the set $L(S) = \{L(\mathbf{u}) : \mathbf{u} \in S\}$ to be the set of images of elements of S. We say the image of the entire set U is the *image* of the function L.

The *kernel* of L is the set $\ker(L) = \{\mathbf{u} \in U : L(\mathbf{u}) = \mathbf{0}\}$ of elements of U whose image is the zero vector.

Another way of thinking about linear transformations is that they send lines to lines. In particular, the image of a subspace under a linear transformation is always a subspace—thus the image of a line will be either a point or a line.

Proposition 4.8. *Let $L : U \to V$ be a linear transformation, and let $S \subseteq U$ be a subspace of U. Then:*

1. $\ker(L)$ *is a subspace of U.*
2. *The image $L(S)$ of S is a subspace of V.*

Proof. 1. See homework 6.

2. We use the subspace theorem:

   (a) We wish to show that $\mathbf{0} \in L(S)$. We claim in particular that $L(\mathbf{0}) = \mathbf{0}$: that is, the image of the zero vector in U must be the zero vector in V. Recall that $0 \cdot \mathbf{v} = \mathbf{0}$ for any $\mathbf{v} \in V$, so we have

   $$L(\mathbf{0}) = L(0 \cdot \mathbf{0}) = 0L(\mathbf{0}) = \mathbf{0}.$$

   Thus since S is a subspace we have $\mathbf{0} \in S$ and thus $\mathbf{0} \in L(S)$.

(b) Suppose $\mathbf{v} \in L(S)$ and $r \in \mathbb{R}$. Then there is some $\mathbf{u} \in S$ with $L(\mathbf{u}) = \mathbf{v}$, and since S is a subspace we know that $r\mathbf{u} \in S$. Thus

$$r\mathbf{v} = rL(\mathbf{u}) = L(r\mathbf{u}) \in L(S).$$

(c) Suppose $\mathbf{v}_1, \mathbf{v}_2 \in L(S)$. Then there exist $\mathbf{u}_1, \mathbf{u}_2 \in S$ such that $L(\mathbf{u}_1) = \mathbf{v}_1$ and $L(\mathbf{u}_2) = \mathbf{v}_2$. Since S is a subspace we know that $\mathbf{u}_1 + \mathbf{u}_2 \in S$. Then

$$\mathbf{v}_1 + \mathbf{v}_2 = L(\mathbf{u}_1) + L(\mathbf{u}_2) = L(\mathbf{u}_1 + \mathbf{u}_2) \in L(S).$$

□

Corollary 4.9. *If $L : U \to V$ is a linear transformation, then the image of L is a subspace of V.*

Example 4.10. In our geometric example of a ninety degree counterclockwise rotation, the kernel is just the origin—nothing gets mapped to the origin except the origin. The image is the entire plane.

Example 4.11. If A is a matrix, then the linear transformation of A has a kernel precisely equal to the nullspace of A, since the nullspace is the set of $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

In section 4.2 we will see that the image of A is the span of the columns of A.

Example 4.12. Let $\mathcal{D}([a, b], \mathbb{R})$ be the space of continuously differentiable functions from the closed interval $[a, b]$ to the real line. Define the derivative operator $D : \mathcal{D}([a, b], \mathbb{R}) \to \mathcal{C}([a, b], \mathbb{R})$ by $D(f) = f'$. First we claim that D is a linear operator: we have that $D(f+g) = (f + g)' = f' + g' = D(f) + D(g)$, and $D(rf) = (rf)' + rf' = rD(f)$.

The kernel of D is the space of constant functions, which is a one-dimensional subspace. The image of D is actually a little hard to see, but it's actually the set of all continuous functions on $[a, b]$.

In other contexts we might write $\frac{d}{dx}$ instead of D for this linear transformation.

Example 4.13. Let $\mathcal{C}([a, b], \mathbb{R})$ be the set of all continuous functions on the closed interval $[a, b]$. The (indefinite) integral isn't quite a linear transformation, since there's an ambiguity in choice of constant. (This is what we mean when we say something is "not well defined": if I tell you to give me the integral of x^2, you can't give me a specific function back so my question is not precise enough).

But the function $I(f) = \int_a^x f(t)\,dt$ is a linear transformation, since $\int_a^x (f + g)(t)\,dt = \int_a^x f(t)\,dt + \int_a^x g(t)\,dt$ and $\int_a^x rf(t)\,dt = r\int_a^x f(t)\,dt$. In this case the choice of a as the basepoint resolves the earlier ambiguity.

The kernel of I is the trivial vector space containing only the zero function. The image is again a bit hard to see, but works out to be the space of differentiable functions with the property that $F(a) = 0$.

This last example shows an important principle: our derivative and integral linear transformations (almost) undo each other. This is a very important property and we will look at it on its own in 4.4.

4.2 Row space, column space and nullspace

We saw that every $m \times n$ matrix gives a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. These linear transformations are easy to work with, because we know a lot about matrices. In this section we'll see what we can learn about those transformations. (In section 4.3 we'll see how to apply this to any linear transformation.)

Definition 4.14. If $A = (a_{ij})$ is a $m \times n$ matrix, then each row can be viewed as a vector in $\mathbb{R}^n$; we call these vectors the *row vectors* of A. We may notate them as $\mathbf{r}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$.

Similarly, we can view each column as vector in $\mathbb{R}^m$, and we call these the *column vectors* of A. We may notate them as $\mathbf{c}_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$

Thus each matrix gives us two sets of vectors. We can look at these vectors and see which vector spaces they span.

Definition 4.15. If A is a $m \times n$ matrix, we say that the span of the row vectors of A is the *row space* of A, which we will sometimes denote $\operatorname{row}(A)$. It is a subspace of $\mathbb{R}^n$. The dimension of the row space is the *rank* of A, denoted $\operatorname{rk}(A)$.

The span of the column vectors of A is the *column space* of A, sometimes denoted $\operatorname{col}(A)$.

Recall that we defined the *nullspace* of A to be the set $N(A) = \ker(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ of solutions to the associated homogeneous system of linear equations. We define the *nullity* of A to be the dimension of $N(A)$.

We want to relate these ideas to the kernel and image we discussed last section. We know that every linear transformation is a matrix, and every matrix is a linear transformation.

It's pretty clear that $N(A)$ is the kernel of the associated linear transformation. By definition, $N(A)$ is the set of $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, and this is just the definition of the kernel of the linear transformation. Thus we sometimes call $N(A)$ the *kernel* of the matrix A as well.

The image is a bit trickier, but still has a clear answer.

Proposition 4.16. *Let A be a $m \times n$ matrix and $\mathbf{b}$ a vector in $\mathbb{R}^m$. Then the image of the linear transformation associated to A is the columnspace of A. That is, $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $col(A)$.*

Proof. The equation $A\mathbf{x} = \mathbf{b}$ is the same as the system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

which we can rewrite as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b}.$$

Thus the equation has a solution precisely when $\mathbf{b}$ is in the span of the $\mathbf{c}_i$, which is the column space of A by definition. □

Corollary 4.17. *The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $b \in \mathbb{R}^m$ if and only if $col(A) = \mathbb{R}^m$, that is, the column vectors span $\mathbb{R}^m$.*

The system has a unique solution if and only if the column vectors are linearly independent.

Proof. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if every $\mathbf{b} \in \mathbb{R}^m$ is in the column space, that is, if the column vectors span $\mathbb{R}^m$.

The column vectors are linearly independent if and only if every vector in their span can be represented uniquely as a linear combination of the column vectors. □

This is only a partial answer, though, since we don't have a way to figure out what the column space actually looks like. To learn about that, we shift to looking at the row space, which is somewhat easier to understand.

Corollary 4.18. *Suppose A is a $m \times n$ matrix and A_R is the matrix obtained by using Gauss-Jordan elimination to reduce it to reduced row echelon form. Then the non-zero rows of A_R form a basis for the row space of A.*

Proof. The non-zero rows of A_R are clearly linearly independent, since each one has a 1 in a column where every other row has a zero. Thus the non-zero rows of A_R form a basis for the space they span, which is the rowspace of A_R. But we saw in section 1.3 that A_R and A have the same rowspace, so clearly the rows of A_R span the rowspace of A. Thus they form a basis for the rowspace of A. □

Example 4.19. Find a basis for the rowspace of $\begin{bmatrix} 1 & 5 & -9 & 11 \\ -2 & -9 & 15 & -21 \\ 3 & 17 & -30 & 36 \\ -1 & 2 & -3 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 5 & -9 & 11 \\ -2 & -9 & 15 & -21 \\ 3 & 17 & -30 & 36 \\ -1 & 2 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -9 & 11 \\ 0 & 1 & -3 & 1 \\ 0 & 2 & -3 & 3 \\ 0 & 7 & -12 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 & 6 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 9 & 3 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 6 & 6 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for row(A) is $\{(1,0,0,4),(0,1,0,2),(0,0,1,1/3)\}$. The matrix has rank 3.

Remark 4.20. We can use this to find a "simple" basis for any vector space we have a spanning set for: write a matrix with our spanning set as rows, and row-reduce it until we have a basis.

As a consequence of this approach to the rowspace, we get an extremely powerful result relating the rank and the nullity:

Theorem 4.21 (Rank-Nullity)**.** *If $A \in M_{m \times n}$ then rank of A plus nullity of A equals n.*

Proof. If U is the reduced row echelon form of A, then $A\mathbf{x} = \mathbf{0}$ is equivalent to $U\mathbf{x} = \mathbf{0}$. Since the matrix has rank r, the matrix U will have r nonzero rows and $n - r$ zero rows; thus it will have $n - r$ free variables and r lead variables.

The dimension of $N(A)$ is equal to the number of free variables, and thus to $n - r$. □

We have managed to relate the rank and the nullity, but we still want to know about the column space. But the column space is tied to the row space in a fundamental way.

Proposition 4.22. *If A is a $m \times n$ matrix, the dimension of the row space of A equals the dimension of the column space of A.*

Proof. We will use a trick with the transpose matrix, since the rows of A are the columns of A^T and vice versa. We will prove that the dimension of the column space of a matrix is at least as great as the dimension of the row space. But since this result will also hold for the transpose matrix, this gives us our answer.

Suppose A has rank r, and let U be the row echelon form of A. It will have r leading 1s, and the columns containing the leading 1s will be linearly independent. (They do not form a basis for the column space, since we have no reason to believe that the row operations preserve the span of the *columns*).

Let U_L be the matrix obtained by deleting the columns of U corresponding to free variables, leaving only the columns that contain a leading 1. Delete the same columns from A, and call the resulting matrix A_L.

The matrices U_L and A_L are row-equivalent, so $A_L\mathbf{x} = \mathbf{0}$ if and only if $U_L\mathbf{x} = \mathbf{0}$, and since the columns of U_L are linearly independent, this happens if and only if $\mathbf{x} = \mathbf{0}$. Thus we see that the columns of A_L are linearly independent. We know that A_L will have exactly r columns, so the column space contains at least r linearly independent vectors, and so the dimension of the column space is at least r. Thus $\dim(col(A)) \geq \dim(row(A)) = r$.

Now consider the matrix A^T. By the previous result, $\dim(\text{col}(A^T)) \geq \dim(\text{row}(A^T))$. But we know that $\text{col}(A^T) = \text{row}(A)$ and $\text{row}(A^T) = \text{col}(A)$, so this tells us that $\dim(\text{row}(A)) \geq \dim(\text{col}(A))$, which combined with the previous result gives us that $\dim(\text{row}(A)) = \dim(\text{col}(A))$.

□

Remark 4.23. This gives us another perspective on our attempts to figure out whether a set of vectors was a spanning set. If we write a matrix whose rows are our vectors, then the rank is the dimension of the rowspace; so the rows span if and only if the rank is equal to the dimension of the space, which is the length of each row, which is the number of columns.

Alternatively, we can make a matrix whose columns are the vectors. Then the rank is the dimension of the columnspace, so our vectors span if and only if the rank is equal to the height of the columns, which is the number of rows.

We still need a way to actually find the image, however. Fortunately, the proof of proposition 4.22 gives us a way.

Corollary 4.24. *Let A be a $m \times n$ matrix, and let U be the reduced row echelon form of A. Then the columns of A corresponding to columns of U that contain a leading "1" form a basis for the columnspace of A.*

Proof. We just showed that these columns are linearly independent, and there are r of them. Thus they are a basis. □

Remark 4.25. Note that the columns of U do not (usually) span the column space of A! But looking at U tells us which columns we should take to find a basis for the column space.

Note that we could also find a basis for the column space by simply taking A^T, row reducing it, and finding a basis for the rowspace of A^T.

Example 4.26. Find a basis for the column space of $\begin{bmatrix} 1 & 5 & -9 & 11 \\ -2 & -9 & 15 & -21 \\ 3 & 17 & -30 & 36 \\ -1 & 2 & -3 & -1 \end{bmatrix}$

We saw that the reduced row echelon form of this matrix has leading ones in the first three columns. So the first three columns form a basis for the columnspace, and thus a basis is $\{(1, -2, 3, -1), (5, -9, 17, 2), (-9, 15, -30, -3)\}$.

Example 4.27. Find bases for the row, column, and nullspace of $\begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$

We first row reduce the matrix.

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 4 & 4 & 12 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 3 & 7 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To find the rowspace, we just take these rows; so a basis for the rowspace is $\{(1, 0, 3, 7, 0), (0, 1, 1, 3, 0), (0, 0, 0, 0, 1)\}$. Thus the rank of the matrix ix 3.

To find the columnspace, we look at the columns corresponding to those with leading 1s, which are the first, second, and fifth. Thus a basis for the columnspace is
$\{(1,-1,0,1),(-2,3,1,2),(2,-2,4,5)\}$.

To find the nullspace, we see there are two free variables, which we set to be parameters $x_3 = \alpha, x_4 = \beta$. Then the nullspace is

$$\begin{aligned}\{(-3\alpha - 7\beta, -\alpha - 3\beta, \alpha, \beta, 0)\} &= \{(-3\alpha, -\alpha, \alpha, 0, 0) + (-7\beta, -3\beta, 0, \beta, 0)\} \\ &= \{\alpha(-3,-1,1,0,0) + \beta(-7,-3,0,1,0)\}\end{aligned}$$

so a basis for the nullspace is $\{(-3,-1,1,0,0),(-7,-3,0,1,0)\}$. The nullity is 2, which is what we expected from the rank-nullity theorem.

4.3 The Matrix of a Linear Transformation

We now know a lot about linear transformations that come from matrices. But what about other linear transformations? It turns out that this isn't really an issue, because it turns out that *all* linear transformations (of finite-dimensional vector spaces) come from matrices.

In essence, we can represent a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ with a matrix because we have a system of coordinates for $\mathbb{R}^n$ and $\mathbb{R}^m$; the matrix tells us what happens to each coordinate.

Example 4.28. Let $A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix}$ be a matrix, and thus a linear transformation $\mathbb{R}^3 \to \mathbb{R}^2$. Let's see what happens to each element of the standard basis for $\mathbb{R}^3$.

$$A\mathbf{e}_1 = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$A\mathbf{e}_1 = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$A\mathbf{e}_1 = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

We notice that the image of the standard basis elements are just the columns of the matrix! This isn't a coincidence; the columns of our matrix are telling us exactly where our basis vectors go.

Proposition 4.29. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$.*

In particular, the jth column vector of A is given by $\mathbf{c}_j = L(\mathbf{e}_j)$.

Proof. According to the theorem statement, we know that $A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{bmatrix}$. So we just need to check that this matrix gives us the linear transformation L.

First we show that our matrix does the right things on the standard basis vectors. We see that

$$A\mathbf{e}_j = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_j & \dots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{c}_j = L(\mathbf{e}_j).$$

Now let $\mathbf{u} \in \mathbb{R}^n$ be any vector. Then we know we can write $\mathbf{u} = \sum_{i=1}^n u_i \mathbf{e}_i$ since every element is some linear combination of basis vectors. Thus we have

$$\begin{aligned} A\mathbf{u} = A\left(\sum_{i=1}^n u_i\mathbf{e}_i\right) = \sum_{i=1}^n Au_i\mathbf{e}_i = \sum_{i=1}^n u_iA\mathbf{e}_i = \sum_{i=1}^n u_iL(\mathbf{e}_i) & \quad \text{by the previous computation} \\ = \sum_{i=1}^n L(u_i\mathbf{e}_i) & \quad \text{scalar multiplication} \\ = L\left(\sum_{i=1}^n u_i\mathbf{e}_i\right) & \quad \text{additivity} \\ = L(\mathbf{u}). & \end{aligned}$$

□

Example 4.30. Let's look at the linear transformation from earlier, of a 90 degree rotation counterclockwise. This is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, so we can find a 2×2 matrix representing it. Let's call the map $R_{\pi/2}$.

By geometry, we see that $R_{\pi/2}(\mathbf{e}_1) = (0, 1) = \mathbf{e}_2$, and that $R_{\pi/2}(\mathbf{e}_2) = (-1, 0) = -\mathbf{e}_1$. Thus the matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0. \end{bmatrix}$

Let's generalize to any rotation; let R_θ be the rotation counterclockwise by θ. To see what happens we have to draw the unit circle; we compute that $R_\theta(\mathbf{e}_1) = (\cos\theta, \sin\theta)$, and $R_\theta(\mathbf{e}_2) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2) = (-\sin(\theta), \cos(\theta)$. Thus the matrix of R_θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Example 4.31. Define a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ by $L(x, y) = (x + y, x - y, 2x)$. First we should check that this is in fact a linear transformation, but I won't do that here.

We need to check the image of $\mathbf{e}_1$ and $\mathbf{e}_2$. We see that

$$L(\mathbf{e}_1) = L(1, 0) = (1, 1, 2)$$
$$L(\mathbf{e}_2) = L(0, 1) = (1, -1, 0).$$

Thus the matrix of L is

$$A_L = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

We can check this by computing

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \\ 2x \end{bmatrix}$$

which is exactly what we should get.

We'd like to be able to do this to any vector space, or at least any finite dimensional one. We need some set of coordinates to let us matricize other linear transformations. Fortunately, we developed those in section 3: a set of coordinates is a basis.

Definition 4.32. If U is a vector space and $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for U, and $\mathbf{u} \in U$, we can write $\mathbf{u} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. We define the *coordinate vector* of $\mathbf{u}$ with respect to E by

$$[\mathbf{u}]_E = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The a_i are called the *coordinates* of $\mathbf{u}$ with respect to the basis E.

We here observe that every $\mathbf{u} \in U$ corresponds to exactly one coordinate vector with respect to E, and vice versa. We will discuss this in more detail in 4.4.

Example 4.33. Let $U = \mathcal{P}_3(x)$. Then $E = \{1, x, x^2, x^3\}$ is a basis for U. Also, $F = \{1, 1 + x, 1 + x^2, 1 + x^3\}$ is a basis for U.

Let $f(x) = 1 + 3x + x^2 - x^3 \in U$. Then

$$[f]_E = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix} \quad [f]_F = \begin{bmatrix} -2 \\ 3 \\ 1 \\ -1 \end{bmatrix}.$$

These are two different vectors of real numbers, but they represent the *same* element of U, just in different bases.

Example 4.34. Let $U = \mathbb{R}^3$ and let $E = \{(1,0,0),(1,1,0),(1,1,1)\}$. Then if $\mathbf{u} = (1,3,2)$, then

$$[\mathbf{u}]_E = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}.$$

Remark 4.35. If B is the standard basis for $\mathbb{R}^n$, then any time we write a column vector there's an implicit $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_B$ that we just don't bother to write down.

Lemma 4.36. *If U is a vector space and $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for U, then the function $[\cdot]_E : U \to \mathbb{R}^n$ which sends $\mathbf{u}$ to $[\mathbf{u}]_E$ is a linear function.*

Proof. Let $\mathbf{u}, \mathbf{v} \in U$ and $r \in \mathbb{R}$. We can write

$$\mathbf{u} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$$
$$\mathbf{v} = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n.$$

Then

$$\begin{aligned} [r\mathbf{u}] &= [ra_1\mathbf{e}_1 + \cdots + ra_n\mathbf{e}_n] = (ra_1, \dots, ra_n) = r(a_1, \dots, a_n) = r[\mathbf{u}]. \\ [\mathbf{u}+\mathbf{v}] &= [(a_1+b_1)\mathbf{e}_1 + \cdots + (a_n+b_n)\mathbf{e}_n] = (a_1+b_1, \dots, a_n+b_n) \\ &= (a_1, \dots, a_n) + (b_1, \dots, b_n) = [\mathbf{u}] + [\mathbf{v}]. \end{aligned}$$

Thus by definition, $[\cdot]_E$ is a linear transformation.

□

Theorem 4.37. *Let U and V be finite-dimensional vector spaces, with $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ a basis for U and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ a basis for V. Let $L : U \to V$ be a linear transformation.*

Then there is a matrix A that represents L with respect to E and F, such that $L\mathbf{u} = \mathbf{v}$ if and only if $A[\mathbf{u}]_E = [\mathbf{v}]_F$. The columns of A are given by $\mathbf{c}_j = [L(\mathbf{e}_j)]_F$.

Remark 4.38. This looks really complicated, but it really just says that any v linear transformation is determined entirely by what it does to the elements of some basis; if you have a basis and you know where your transformation sends each element of that basis, you know what it does to everything in your space.

In particular, if we have coordintes for our vector spaces, we can use a matrix to map one set of coordinates to the other, as if we were working in $\mathbb{R}^n$.

$$\begin{array}{ccc} U & \xrightarrow{L} & V \\ \downarrow{\scriptstyle [\cdot]_E} & & \downarrow{\scriptstyle [\cdot]_F} \\ \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \end{array} \qquad \begin{array}{ccc} \mathbf{u} & \overset{L}{\longmapsto} & L(\mathbf{u}) \\ \downarrow{\scriptstyle [\cdot]_E} & & \downarrow{\scriptstyle [\cdot]_F} \\ [\mathbf{u}]_E & \overset{A}{\longmapsto} & A[\mathbf{u}]_E = [L(\mathbf{u})]_F \end{array}$$

Proof. We just want to show that $A[\mathbf{u}]_E = [L(\mathbf{u})]_F$ for any $\mathbf{u} \in U$, where

$$A = [\mathbf{c}_1 \dots \mathbf{c}_n] = [[L(\mathbf{e}_1)]_F \dots [L(\mathbf{e}_n)]_F] .$$

Our proof is essentially the same as the proof of Proposition 4.29. Let $\mathbf{u} \in U$. Since E is a basis for U we can write $u = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. Then we have

$$\begin{aligned} [L(\mathbf{u})]_F &= [a_1 L(\mathbf{e}_1) + \cdots + a_n L(\mathbf{e}_n)]_F = a_1 [L(\mathbf{e}_1)]_F + \cdots + a_n [L(\mathbf{e}_n)]_F \\ &= a_1\mathbf{c}_1 + \cdots + a_n\mathbf{c}_n; \\ A[\mathbf{u}]_E &= A [a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n]_E = A(a_1, \dots, a_n) = [\mathbf{c}_1 \dots \mathbf{c}_n] (a_1, \dots, a_n) \\ &= \mathbf{c}_1 a_1 + \cdots + \mathbf{c}_n a_n. \end{aligned}$$

Thus we have $[L(\mathbf{u})]_F = A [\mathbf{u}]_E$, so the matrix A does in fact represent the linear operator L. □

Example 4.39. Let $F = \{(1,1), (-1,1)\}$ be a basis for $\mathbb{R}^2$, and let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $L(x,y,z) = (x-y-z, x+y+z)$. Find a matrix for L with respect to the standard basis in the domain and F in the codomain.

$$\begin{aligned} L(1,0,0) &= (1,1) = \mathbf{f}_1 \\ L(0,1,0) &= (-1,1) = \mathbf{f}_2 \\ L(0,0,1) &= (-1,1) = \mathbf{f}_2 \\ A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Example 4.40. Let S be the subspace of $\mathcal{C}([a,b],\mathbb{R})$ spanned by $\{e^x, xe^x, x^2e^x\}$, and let D be the differentiation operator on S. Find the matrix of D with respect to $\{e^x, xe^x, x^2e^x\}$.

We compute:

$$\begin{aligned} D(e^x) &= e^x = \mathbf{s}_1 \\ D(xe^x) &= e^x + xe^x = \mathbf{s}_1 + \mathbf{s}_2 \\ D(x^2e^x) &= 2xe^x + x^2e^x = 2\mathbf{s}_2 + \mathbf{s}_3 \\ A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Example 4.41. Let $E = \{(1,1,0),(1,0,1),(0,1,1)\}$ and $F = \{(1,0,0),(1,1,0),(1,1,1)\}$ be bases for $\mathbb{R}^3$, and define $L(x,y,z) = (x+y+z, 2z, -x+y+z)$. We can check this is a linear transformation.

To find the matrix of L with respect to E and the standard basis, we compute

$$\begin{aligned} L(1,1,0) &= (2,0,0) \\ L(1,0,1) &= (2,2,0) \\ L(0,1,1) &= (2,2,2). \end{aligned}$$

Thus the matrix with respect to E and the standard basis is

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

If we want to find the matrix with respect to E and F, we observe that

$$\begin{aligned} L(1,1,0) &= (2,0,0) = 2(1,0,0) = 2\mathbf{f}_1 \\ L(1,0,1) &= (2,2,0) = 2(1,1,0) = 2\mathbf{f}_2 \\ L(0,1,1) &= (2,2,2) = 2(1,1,1) = 2\mathbf{f}_3. \end{aligned}$$

Thus the matrix is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We notice that this matrix is really simple; this is a "good" choice of bases for this linear transformation.

In contrast, let's look at the transformation $T(x, y, z) = (x, y, z)$. Then we have

$$\begin{aligned} T(1,1,0) &= (1,1,0) = (1,1,0) = \mathbf{f}_2 \\ T(1,0,1) &= (1,0,1) = (1,0,0) - (1,1,0) + (1,1,1) = \mathbf{f}_1 - \mathbf{f}_2 + \mathbf{f}_3 \\ T(0,1,1) &= (0,1,1) = -(1,0,0) + (1,1,1) = -\mathbf{f}_1 + \mathbf{f}_3. \end{aligned}$$

Thus the matrix of T with respect to E and F is

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Thus this transformation, which is really simple with respect to the standard basis, is much more complicated with respect to these bases.

We'll talk a lot more about this choice of basis idea in section 6, and we'll talk about how to find the best basis for a given linear transformation in section 5.

This allows us to translate most of the results from section 4.2 to tell us about any linear transformation of vector spaces. The image of a transformation corresponds to the columnspace of the associated matrix; the kernel corresponds to the nullspace of the associated matrix.

Then this gives us a generalization of the rank-nullity theorem: the rank is the dimension of the rowspace, which is the dimension of the columnspace, which is the dimension of the image. And the nullity is the dimension of the nullspace, which is the dimension of the kernel. The rank-nullity theorem 4.21 tells us the rank and nullity add up to the number of columns of the associated matrix—which is the dimension of the domain of the linear transformation. All combined, this gives us

Theorem 4.42 (Rank-Nullity for Vector Spaces)**.** *Let U, V be finite-dimensional vector spaces, and $L : U \to V$ be a linear transformation. Then* $\dim \ker(L) + \dim \operatorname{Im}(L) = \dim U$.

Example 4.43. Define $L : \mathcal{P}_3(x) \to \mathcal{P}_3(x)$ be given by $L(f) = (1 + x)f'' - f'$. We can take a standard basis $\{(1, x, x^2, x^3)\}$ and compute the matrix:

$$\begin{aligned} L(1) &= 0 & L(x) &= -1 \\ L(x^2) &= (1+x)2 - 2x = 2 & L(x^3) &= (1+x)6x - 3x^2 = 6x + 3x^2 \end{aligned}$$

so we get the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is already pretty close to row-reduced, but we can finish it off and get the reduction

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has rank 2 and nullity 2. Thus the dimension of the image is 2; it's spanned by the second and fourth columns of the original matrix, which correspond to $\{-1, 6x + 3x^2\}$. And the dimension of the kernel is 2. The kernel satisfies $a_1 + a_2 = 0, a_3 = 0$, so it's the set $\{a_0 + a_1 x - a_1 x^2\}$.

As a final result, we will see that we now know about every possible subspace. We know that the kernel of a linear transformation is a subspace; but the converse is true as well, and every subspace is the kernel of some linear transformation.

Proposition 4.44. *Let V be a vector space and $U \subset V$ a subspace. Then U is the kernel of some linear transformation.*

Proof. We'll prove this in the case where U and V are finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for U. By basis padding, there is a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}_1, \dots, \mathbf{v}_m\}$ for the vector space V.

Define a linear transformation $L : V \to V$ by setting $L(\mathbf{u}_i) = \mathbf{0}$ and $L(\mathbf{v}_i) = \mathbf{v}_i$. That is, for any $\mathbf{v} \in V$, we can write

$$v = a_1 \mathbf{u}_1 + \cdots + a_n \mathbf{u}_n + b_1 \mathbf{v}_1 + \cdots + b_m \mathbf{v}_m,$$

so we define

$$L(\mathbf{v}) = b_1 \mathbf{v}_1 + \cdots + b_m \mathbf{v}_m.$$

Then the kernel of L is the set spanned by $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$, which is just U. □

4.4 Isomorphisms

In the previous section, we looked at a lot of linear transformations. In particular, we saw the coordinate map $[\cdot]_E : U \to \mathbb{R}^n$ which sends a vector $\mathbf{u} \in U$ to its coordinates with respect to E. We observed that this mapping in fact goes both ways: if we have a vector we can compute the coordinates, and if we have coordinates we can compute the vectors. Functions like this are very important and have a special name.

Definition 4.45. Let $f : U \to V$ be a function. If there is a $g : V \to U$ such that $g(f(\mathbf{u})) = \mathbf{u}$ for all $u \in U$, and $f(g(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v} \in V$, then we say that $g = f^{-1}$ is the *inverse* of f, and that f is *invertible*.

If f is an invertible linear transformation, we say that f is an *isomorphism* between U and V.

If U and V are vector spaces, we say they are *isomorphic* if there exists an isomorphism from U to V. We write $U \cong V$.

Remark 4.46. We will see that if two spaces are isomorphic, we can treat them as essentially the same. This does not mean they are the same; $\mathbb{R}^5$ is not the same thing as the space of degree-four polynomials.

But if $U \cong V$ then they are the same *as vector spaces*, because if we want to do something to U, we can instead map it to V, do it there, and then map it back.

Example 4.47. Let U be a vector space with basis $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, and let $f : U \to \mathbb{R}^n$ be defined by $f(\mathbf{u}) = [\mathbf{u}]_E$. Then f is invertible, and the inverse of f is given by the function $g(a_1, \dots, a_n) = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. To prove this, we check two things.

For any $\mathbf{u} \in U$ we can write $\mathbf{u} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. Then

$$g(f(\mathbf{u})) = g(f(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n)) = g(a_1, \dots, a_n) = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n = \mathbf{u}.$$

Similarly, for any $(a_1, \dots, a_n) \in \mathbb{R}^n$ we have

$$f(g(a_1, \dots, a_n)) = f(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = (a_1, \dots, a_n).$$

Thus g is the inverse of f by definition.

Example 4.48. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x+y, x-y)$. Then f is invertible, with inverse $g(a, b) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right)$. To prove this we check:

$$g(f(x,y)) = g(x+y, x-y) = \left(\frac{(x+y)+(x-y)}{2}, \frac{(x+y)-(x-y)}{2}\right) = (x,y)$$
$$f(g(a,b)) = f\left(\frac{a+b}{2}, \frac{a-b}{2}\right) = \left(\frac{a+b}{2} + \frac{a-b}{2}, \frac{a+b}{2} - \frac{a-b}{2}\right) = (a,b).$$

Thus g is the inverse of f by definition.

So far we can check whether a given g is the inverse of f, but we don't have a good way of determining if a function is invertible. In order to do that, we have to recall a few properties of functions. (Discrete Math covers these topics in much more detail; we need much less knowledge and information about them).

Definition 4.49.

- A function f is *one-to-one* or *injective* if it has the property that: if $f(x) = f(y)$ then $x = y$. This tells us that anything in the image of f is only in the image once.
- A function $f : A \to B$ is *onto* or *surjective* if the image of f is B. That is, f is onto if for every $b \in B$ there is an $a \in A$ with $f(a) = b$. This tells us we can reach every element of the codomain from some element of the domain.
- A function f is *bijective* if it is both one-to-one and onto.

Proposition 4.50. *Let $L : U \to V$ be a linear transformation of vector spaces. Then:*

1. *L is one-to-one if and only if* $\ker(L) = \{\mathbf{0}\}$.
2. *L is invertible if and only if L is bijective.*

Proof.

1. See Homework 8.
2. Suppose L is bijective. Define a transformation $T : V \to U$ as follows: let $\mathbf{v} \in V$. Then L is onto, so by definition there is some $\mathbf{u} \in U$ such that $L(\mathbf{u}) = \mathbf{v}$. Since L is one-to-one there is only one such element, since if $L(\mathbf{u}) = L(\mathbf{u}_1)$ then $\mathbf{u} = \mathbf{u}_1$ by definition of one-to-one. Define $T(\mathbf{v}) = \mathbf{u}$.

   Then for any $\mathbf{v} \in V$, we have $L(T(\mathbf{v})) = L(\mathbf{u}) = \mathbf{v}$, and for any $\mathbf{u} \in U$ we have $T(L(\mathbf{u})) = T(\mathbf{v}) = \mathbf{u}$. Thus by definition, $T = L^{-1}$.

   Conversely, suppose L is invertible, and let $T = L^{-1}$. Suppose $L(\mathbf{u}) = L(\mathbf{v})$. Then $T(L(\mathbf{u})) = T(L(\mathbf{v}))$, but $T(L(\mathbf{u})) = \mathbf{u}$ and $T(L(\mathbf{v})) = \mathbf{v}$, so $\mathbf{u} = \mathbf{v}$, and by definition L is one-to-one.

   Let $\mathbf{v} \in V$. Then $T(\mathbf{v}) \in U$, and $L(T(\mathbf{v})) = \mathbf{v}$, so $\mathbf{v} \in L(U)$ for any $\mathbf{v}\ inV$. Thus L is onto by definition, and since it is one-to-one and onto, it is bijective.

□

Remark 4.51. Another way to think of the second result is that "onto" guarantees that every element $\mathbf{v} \in V$ has an inverse, and "one-to-one" guarantees that no element has more than one, so the inverse function is actually well-defined as a function.

We can contrast with, say, the function $f(x) = x^2$. This function is not one-to-one, so when we ask for the inverse or square root of 4, we get two possible answers. (This function isn't linear, but we didn't actually use linearity anywhere in the previous proof, and in fact it works for all functions).

Proposition 4.50 gives us an easy way to check if a linear function is injective, and if we also check that it is surjective we can easily see whether it is invertible. We can always check surjectivity directly, and on your homework you will, but we'd like to make this easier as well. To do that we want to convert the Rank-Nullity Theorem to discuss all linear transformations. First we need to lay some groundwork.

Recall the rank-nullity thoerem:

Theorem 4.52 (Rank-Nullity Theorem)**.** *If* U, V *are finite-dimensional, then* $\dim U = \dim \ker(L) + \dim L(U)$*.*

Corollary 4.53. *Let* $L : U \to V$ *be linear. Then* L *is one-to-one if and only if* $\dim U = \dim L(U)$*, and* L *is onto if and only if* $\dim V = \dim(U) - \dim \ker(L)$*.*

If $\dim U = \dim V$*,* L *is an isomorphism if and only if* $\ker(L) = \{\mathbf{0}\}$*.*

Now we can easily determine if a transformation is invertible. But how do we find the inverse? Like everything in linear algebra, it's easier to do computations if we change things to be a matrix.

Proposition 4.54. *Let* $f : U \to V$ *be a linear transformation of finite dimensional vector spaces, and let* E, F *be bases for* U, V *respectively. Let* A *be the matrix of* f *with respect to* E, F*. Then* f *is invertible if and only if* A *is invertible, and the matrix of* f^{-1} *is* A^{-1}*.*

Proof. Suppose f is invertible, and that the matrix of f is A and the Let B be the matrix of f^{-1}. Then for any $\mathbf{u} \in U$,

$$[f^{-1}(f(\mathbf{u}))]_E = B[f(\mathbf{u})]_F = BA[\mathbf{u}]_E$$
$$[f^{-1}(f(\mathbf{u}))]_E = [\mathbf{u}]_E$$

and thus $BA[\mathbf{u}]_E = [\mathbf{u}]_E$ for all $\mathbf{u} \in U$. Thus $BA\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, and thus $BA = I_n$. So by definition $B = A^{-1}$.

Conversely, suppose the matrix of f is A, and A has an inverse A^{-1}. Let g be the function corresponding to A^{-1}, so for all $\mathbf{v} \in V$ we have $[g(\mathbf{v})]_E = A^{-1}[\mathbf{v}]_F$. Then for any $\mathbf{u} \in U, \mathbf{v} \in V$, we compute

$$[g(f(\mathbf{u}))]_E = A^{-1}[f(\mathbf{u})]_F = A^{-1}A[\mathbf{u}]_E = [\mathbf{u}]_E$$
$$[f(g(\mathbf{v}))]_F = A[g(\mathbf{v})]_E = AA^{-1}[\mathbf{v}]_F = [\mathbf{v}]_F.$$

Thus $g(f(\mathbf{u})) = \mathbf{u}$ and $f(g(\mathbf{v})) = \mathbf{v}$, so by definition $g = f^{-1}$. □

Corollary 4.55. *A $n \times n$ matrix is invertible if and only if its nullspace is trivial.*

This gives us a method for finding the inverse of any linear transformation: we find the matrix of the transformation, use Gaussian elimination to invert the matrix, and then return to the corresponding transformation.

Example 4.56. Let $L(x, y, z) = (x + y + z, 2x + 3y + 2z, x + 5y + 4z)$. What is L^{-1}?

The matrix for L (with respect to the standard basis) is $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 5 & 4 \end{bmatrix}$. So we compute

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 3 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 3 & 7 & -4 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/3 & -4/3 & 1/3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7/3 & -4/3 & 1/3 \end{array}\right]$$

Since A is invertible, this tells us that L is also invertible. And from A^{-1} we can see that

$$L^{-1}(a, b, c) = (2a/3 + b/3 - c/3, -2a + b, 7a/3 - 4b/3 + c/3).$$

We can check by multiplying the original matrices together and seeing that we get the identity, or by computing $L^{-1}(L(x, y, z))$ and confirming that we get $(x, y, z$.

Example 4.57. Let $E : \mathcal{P}_2(x) \to \mathbb{R}^3$ be given by $E(f(x)) = (f(-1), f(0), f(1))$. Can we find an inverse for this function?

Let $\{1, x, x^2\}$ be the basis for $\mathcal{P}_2(x)$, and use the standard basis for $\mathbb{R}^3$. Then the matrix of this transformation is $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. To find the inverse, we compute

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right].$$

We can double-check our work, again, by multiplying out the original matrices.

What have we concluded? If we have a quadratic polynomial such that $f(-1) = a, f(0) = b, f(1) = 3$, then we must have

$$f(x) = b + (c/2 - a/2)x + (a/2 + c/2 - b)x^2.$$

Thus we can use this technique to find the (minimal degree) polynomial that goes through a given set of points.

Example 4.58. Let $D : \mathcal{P}_3(x) \to \mathcal{P}_3(x)$ be given by the derivative map. Is this function invertible?

The function is not invertible, since it has non-trivial kernel. We can also see this by writing down the matrix relative to the obvious basis: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since there is a row of all zeroes, the rows are not linearly independent, so the matrix is not invertible.

Let's tweak things a bit. Let $Q = \{ax + bx^2 + cx^3\} \subset \mathcal{P}_3(x)$, and let $D : Q \to \mathcal{P}_2(x)$ be given by the derivative. Then if we let $E = \{x, x^2, x^3\}$, $F = \{1, x, x^2\}$ be bases for the domain and codomain, we see the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ whose inverse is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Thus this function is invertible; in fact we see the inverse sends $a + bx + cx^2 \mapsto ax + bx^2/2 + cx^3/3$, which you should recognize as an integral.

As a final coda, we'll note that every vector space is isomorphic to itself. This fact will be really important when we want to change coordinate systems.

Proposition 4.59. *Let V be a vector space. Then $V \cong V$. In particular, the identity map Id_V defined by $Id_V(\mathbf{v}) = \mathbf{v}$ is an isomorphism from V to V.*

But two isomorphic vector spaces can have more than one isomorphism between them. In fact, any non-trivial vector space has infinitely many isomorphisms from itself to itself, and these isomorphisms are extremely useful.

Proposition 4.60. *Let U, V be vector spaces, let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for U, and $L : U \to V$ be a linear map. Then $L(E)$ spans $L(U)$, and L is an isomorphism if and only if $L(E)$ is a basis for V.*

Corollary 4.61. *Let U, V be vector spaces, with $\dim U = \dim V$. Then $U \cong V$.*

Proof. Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for U, and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ a basis for V; we know they have the same number of elements since $\dim U = \dim V$. Define a linear map $L : U \to V$ by linearly extending $L(\mathbf{e}_i) = \mathbf{f}_i$; that is, we define

$$L(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = a_1\mathbf{f}_1 + \cdots + a_n\mathbf{f}_n.$$

This map sends a basis to a basis, and thus is an isomorphism. □

Corollary 4.62. *A linear map from V to V is an isomorphism if and only if it sends a basis to a basis.*

5 Eigenvectors and Eigenvalues

In this section we will study a special type of basis, called an eigenbasis. For (almost) any given operator, we get a specific basis which will make most our computations easier.

5.1 Eigenvectors

Definition 5.1. Let $L : V \to V$ be a linear transformation, and let λ be a scalar. If there is a vector $\mathbf{v} \in V$ such that $L\mathbf{v} = \lambda\mathbf{v}$, then we say that λ is an *eigenvalue* of L, and $\mathbf{v}$ is an *eigenvector* with eigenvalue λ.

Geometrically, an eigenvector corresponds to a direction in which our linear operator purely stretches or shrinks vectors, without rotating or reflecting them at all. It can often be an axis of rotation.

Example 5.2. Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. We can check that if $\mathbf{x} = (2, 1)$, then

$$A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

so $\mathbf{x}$ is an eigenvector with eigenvalue 3. Similarly, we can check that if $\mathbf{y} = (1, 1)$, then

$$A\mathbf{y} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus $\mathbf{y}$ is an eigenvector with eigenvalue 2.

Example 5.3. Let $R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation map. We can see geometrically that this has no non-trivial eigenvectors, since it changes the direction of any vector. Algebraically, if (x, y) is an eigenvector, then we would have

$$R_{\pi/2}(x, y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

and thus weh ave $\lambda y = x, \lambda x = -y$, and the only solution here is $x = y = 0$.

In contrast, if we take the rotation map $R : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates around the z-axis, the vector $(0, 0, 1)$ will be an eigenvector with eigenvalue 1.

Example 5.4. Let $V = \mathcal{D}(\mathbb{R}, \mathbb{R})$ be the space of differentiable real functions, and let $\frac{d}{dx} : V \to V$ be the derivative map. If $f(x) = e^{rx}$, then $\frac{d}{dx} f(x) = re^{rx} = rf(x)$, so f is an eigenvector with eigenvalue r.

Proposition 5.5. *Let V be a vector space and $L : V \to V$ a linear transformation. $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if $\mathbf{v} \in \ker(L - \lambda I)$.*

Proof. $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if $L\mathbf{v} = \lambda\mathbf{v} = \lambda I\mathbf{v}$, if and only if $\mathbf{0} = L\mathbf{v} - \lambda I\mathbf{v} = (L - \lambda I)\mathbf{v}$, if and only if $\mathbf{v} \in \ker(L - \lambda I)$. □

Corollary 5.6. *The set of eigenvectors with eigenvalue λ is a subspace of V, called the* eigenspace *corresponding to λ. We denote this space E_λ.*

Corollary 5.7. *A transformation L is invertible if and only if 0 is not an eigenvalue of L.*

Proposition 5.8. *Let $L : V \to V$ be a linear transformation. If $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a set of eigenvectors each with a distinct eigenvalue, then E is linearly independent.*

Proof. Let λ_i be the eigenvalue corresponding to $\mathbf{e}_i$. Suppose (for contradiction) that E is linearly dependent, and let k be the smallest positive integer such that $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ is linearly dependent; then we must have $a_k \neq 0$, and we can compute

$$\begin{aligned}
\mathbf{e}_k &= \frac{-a_1}{a_k}\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}\mathbf{e}_{k-1} \\
L(\mathbf{e}_k) &= L\left(\frac{-a_1}{a_k}\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}\mathbf{e}_{k-1}\right) = \frac{-a_1}{a_k}L(\mathbf{e}_1) + \cdots + \frac{-a_{k-1}}{a_k}L(\mathbf{e}_{k-1}) \\
\lambda_k\mathbf{e}_k &= \frac{-a_1}{a_k}\lambda_1\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}\lambda_{k-1}\mathbf{e}_{k-1}.
\end{aligned}$$

We can multiply the first equation by λ_1 and subtract from the last equation; this gives us

$$\mathbf{0} = \frac{-a_1}{a_k}(\lambda_1 - \lambda_k)\mathbf{e}_1 + \cdots + \frac{-a_{k-1}}{a_k}(\lambda_{k-1} - \lambda_k)\mathbf{e}_{k-1}.$$

But we know by hypothesis that the set $\{\mathbf{e}_1, \dots, \mathbf{e}_{k-1}\}$ is linearly independent, so all these coefficients must be zero. Since the a_i are not all zero, we must have at least some $\lambda_i - \lambda_k = 0$. □

It's straightforward enough to *check* that a vector is an eigenvector if we already have a candidate; but how do we find them? Sometimes this is easy

Example 5.9. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. What are the eigenvalues and eigenspaces of A?

We see that

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}.$$

Thus the eigenvalues are 3 and 2; the corresponding eigenspaces are spanned by $(1, 0)$ and $(0, 1)$, respectively.

When things aren't this easy, there is still a fairly straightforward approach we can take:

Example 5.10. Let $B = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}$. Find the eigenvalues and eigenvectors of B.

If $\mathbf{x} = (x, y)$ is an eigenvector with eigenvalue λ, then we have

$$B\mathbf{x} = \begin{bmatrix} 7x + 2y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

so we have the system of equations $7x + 2y = \lambda x, 3x + 8y = \lambda y$. Equivalently, we have $(7 - \lambda)x + 2y = 0$ and $(3x + (8 - \lambda)y = 0$. We row-reduce

$$\begin{bmatrix} 7-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} \to \begin{bmatrix} 3 & 8-\lambda \\ 0 & 2 + (8-\lambda)(\lambda-7)/3 \end{bmatrix}$$
$$\to \begin{bmatrix} 3 & 8-\lambda \\ 0 & 6 + (-56 + 15\lambda - \lambda^2) \end{bmatrix} = \begin{bmatrix} 3 & 8-\lambda \\ 0 & -\lambda^2 + 15\lambda - 50 \end{bmatrix}.$$

We first see that this is solveable if and only if $0 = \lambda^2 - 15\lambda + 50 = (\lambda - 5)(\lambda - 10)$, and thus if $\lambda = 5$ or $\lambda = 10$. Thus these are the two eigenvalues for B.

If $\lambda = 5$ then we have $3x + 3y = 0$ so $y = -x$. Any vector $(\alpha, -\alpha)$ will be an eigenvector with eigenvalue 5, so the eigenspace for 5 is the span of $\{(1, -1)\}$. And indeed, we compute

$$B(1, -1) = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If $\lambda = 10$ then we have $3x - 2y = 0$ and $y = 3/2x$. Thus any vector $(2\alpha, 3\alpha)$ will be an eigenvector with eigenvalue 10, and the corresponding eigenspace is spanned by $\{(2, 3)\}$. We check:

$$B(2, 3) = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

As the previous example shows, it is completely possible to find the eigenvectors and eigenvalues with the tools we have already, but it's pretty fiddly even for a small example. We'd like to streamline the process, and this leads us to define the determinant.

5.2 Determinants

Definition 5.11. Let $A \in M_{n \times n}$. If A has n distinct eigenvalues, we say that the *determinant* of A, written $\det A$, is the product of the eigenvalues.

More generally, the determinant of A is the product of the eigenvalues "up to multiplicity". Thus if the eigenspace of $\lambda = 2$ is three-dimensional, we will multiply in λ three times.

Definition 5.12 (Formal definition we won't really use)**.**

$$\det A = \prod_{\lambda} \lambda^{e_\lambda} \qquad \text{where} \quad e_\lambda = \dim \ker(A - \lambda I)^n.$$

The determinant is (roughly) the product of the eigenvalues, so it can tell something about what the eigenvalues are. But this doesn't help if we don't have a way of finding the determinant without already knowing the eigenvalues. Fortunately, there is a simple way to compute it.

Example 5.13. The determinant of $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ is $3 \cdot 2 = 6$.

The determinant of $B = \begin{bmatrix} 7 & 2 \\ 3 & 8 \end{bmatrix}$ is $5 \cdot 10 = 50$.

Geometrically, the determinant represents the volume of the n-dimensional solid that our matrix sends the n-dimensional unit cube to; thus it tells us how much our matrix stretches its inputs.

5.2.1 The Laplace Formula

We first need to develop some notation.

Definition 5.14. Let $A = (a_{ij})$ be a $n \times n$ matrix. We define the i, j*th minor matrix* of A to be the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting the row and column containing a_{ij}—that is, deleting the ith row and jth column.

We define the i, j*th minor* of A to be $\det M_{ij}$. We define the i, j*th cofactor* to be $A_{ij} = (-1)^{i+j} \det(M_{ij})$.

Example 5.15. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix}.$$

Then we have

$$M_{1,1} = \begin{bmatrix} -2 & -1 \\ 3 & 3 \end{bmatrix} \qquad M_{3,2} = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}.$$

Fact 5.16 (Cofactor Expansion). *Let A be a $n \times n$ matrix.*

If $A \in M_{1\times 1}$ then $A = \begin{bmatrix} a_{11} \end{bmatrix}$ and $\det A = a_{11}$.

Otherwise, for any k we have

$$\begin{aligned}\det(A) &= \sum_{i=1}^{n} a_{ki}A_{ki} = a_{k1}A_{k1} + a_{k2}A_{k2} + \cdots + a_{kn}A_{kn} \\ &= \sum_{i=1}^{n} a_{ik}A_{ik} = a_{1k}A_{1k} + a_{2k}A_{2k} + \cdots + a_{nk}A_{nk}.\end{aligned}$$

Thus we may compute the determinant of a matrix inductively, using cofactor expansion. We can expand along any row or column; we should pick the one that makes our job easiest.

Remark 5.17. This is usually taken to be the definition of determinant. Feel free to think of it that way, and the fact about eigenvectors as a theorem.

You can also think of the determinant as the unique multilinear map that satisfies certain properties. You probably shouldn't, at the moment. But you can.

Example 5.18. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. If we expand along the last row, we get

$$\begin{aligned}\det A &= 0 \cdot (-1)^{3+1} \det \begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix} + 0 \cdot (-1)^{3+2} \det \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} + 2 \cdot (-1)^{3+3} \det \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix} \\ &= 2 \det \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix} = 2\left(0 \cdot (-1)^{2+1} \det \begin{bmatrix} 2 \end{bmatrix} + 5 \cdot (-1)^{2+2} \det \begin{bmatrix} 3 \end{bmatrix}\right) \\ &= 2(0 + 5 \cdot 3) = 30.\end{aligned}$$

Example 5.19. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & -2 & -1 \\ 3 & 3 & 3 \end{bmatrix}.$$

We'd like to expand along the row or column wiht the most zeros, but we don't have any. I'm going to expand along the bottom row because at least everything is the same.

$$\begin{aligned}\det A &= 3(-1)^{3+1} \det \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} + 3(-1)^{3+2} \det \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} + 3(-1)^{3+3} \begin{bmatrix} 3 & 1 \\ 5 & -2 \end{bmatrix} \\ &= 3\left(1(-1)^{1+1}(-1) + 2(-1)^{1+2}(-2)\right) - 3\left(3(-1)^{1+1}(-1) + 2(-1)^{1+2}5\right) \\ &\qquad + 3\left(3(-1)^{1+1}(-2) + 1(-1)^{1+2}(5)\right) \\ &= 3(-1+4) - 3(-3-10) + 3(-6-5) = 9 + 39 - 33 = 15.\end{aligned}$$

Using this method, we can compute the determinant of any size of matrix. But for small matrices we can work out quick formulas that encode all this information.

Proposition 5.20.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

5.2.2 Properties of Determinants

We'd like to do things to make computing determinants easier, in addition to the formulas I just gave. We can start by proving some simple results.

Proposition 5.21. *If A is a $n \times n$ triangular matrix, then $\det A$ is the product of the diagonal entries of A.*

Proof. We use cofactor expansion; at each step, we have a row or column with only one non-zero entry, on the diagonal. At the end of the cofactor expansion we have simply taken the product of the diagonal entries. □

Proposition 5.22. *If A has a row or column of all zeroes, then $\det A = 0$.*

Proof. Do cofactor expansion along the row of all zeros. □

Proposition 5.23. $\det A^T = \det A$.

Proof. Do a cofactor expansion along the column of A^T that corresponds to the row you expanded along in A, or vice versa. □

Fact 5.24 (Row Operations)**.**

- *Interchanging two rows multiplies the determinant by -1.*
- *Multiplying a row by a scalar multiplies the determinant by that scalar.*
- *Adding a multiple of one row to another row does not change the determinant.*
- $$\det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix} + \det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{b}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{a}_i + \mathbf{b}_i \\ \vdots \\ \mathbf{r}_n \end{bmatrix}.$$

Proof. The proof is really tedious and just involves a bunch of inductions on cofactor expansions. □

Example 5.25.

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 1 \qquad \det \begin{bmatrix} 3 & 3 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 3$$

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \qquad \det \begin{bmatrix} 4 & 4 & 4 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = 3 + 1 = 4.$$

Corollary 5.26. $\det A = 0$ *if and only if the rows of A are linearly dependent.*

Proposition 5.27. *A matrix A is invertible if and only if* $\det A \neq 0$.

Proof. We can view this proof in two different ways.

From the eigenvalue perspective: $\det A$ is the product of the eigenvalues. Thus $\det A = 0$ if and only if 0 is an eigenvalue of A. But 0 is an eigenvalue of A if and only if A has non-trivial kernel, and A is invertible if and only if $\ker(A)$ is trivial.

From the cofactor perspective: if A is invertible it is row-equivalent to the identity matrix, which has determinant 1. None of the row operations can change a determinant from zero to non-zero or vice versa, so $\det A$ is nonzero.

Conversely, if A is not invertible, it is row-equivalent to a matrix with a row of all zeros, which has determinant zero. Since row operations cannot change a determinant from non-zero to zero, $\det A = 0$ as well. □

Fact 5.28. *If A, B are $n \times n$ matrices, then* $\det(AB) = \det(A)\det(B)$.

Corollary 5.29. *If A is a nonsingular matrix, then* $\det(A^{-1}) = \frac{1}{\det A}$.

Remark 5.30. This is why the inverse of a matrix so often has the same denominator appearing in most of the entries; it's the reciprocal of the determinant.

Example 5.31. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

We check this by multiplying the two of them:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

5.3 Characteristic Polynomials

Definition 5.32. We say that $\chi_A(\lambda) = \det(A - \lambda I)$ is the *characteristic polynomial* of A. This is a polynomial in one variable, λ. We call the equation $\chi_A(\lambda) = 0$ the *characteristic equation* of A.

Proposition 5.33. *The real number λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A. That is, the roots of $\chi_A(\lambda)$ is the set of eigenvalues of A.*

Proof. Recall that $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if $\mathbf{v} \in \ker(A - \lambda I)$. Thus λ is an eigenvalue if and only if $\ker(A - \lambda I)$ has nontrivial kernel, which occurs if and only if $\det(A - \lambda I) = 0$. □

Example 5.34. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$.

The characteristic equation is

$$\begin{aligned} 0 = \chi_A(\lambda) &= \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} \\ &= (3-\lambda)(-2-\lambda) - 2 \cdot 3 = -6 - 3\lambda + 2\lambda + \lambda^2 - 6 \\ &= \lambda^2 - \lambda - 12 = (\lambda - 4)(\lambda + 3) \end{aligned}$$

so the eigenvalues are 4 and -3. We compute

$$A - 4I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

so $\ker(A - 4I) = \{\alpha(2,1)\}$. Thus the eigenspace corresponding to 4 is $E_4 = \operatorname{span}\{(2,1)\}$. Similarly,

$$A + 3I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

so $\ker(A + 3I) = \{\alpha(-1,3)\}$. Thus the eigenspace $E_{-3} = \operatorname{span}\{(-1,3)\}$.

Example 5.35. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$.

The characteristic equation is

$$\begin{aligned} 0 = \chi_A(\lambda) &= \left| \begin{pmatrix} 5-\lambda & 1 \\ 3 & 3-\lambda \end{pmatrix} \right| \\ &= (3-\lambda)(5-\lambda) - 1 \cdot 3 = 15 - 8\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2) \end{aligned}$$

so the eigenvalues are 6 and 2.

$$A - 6I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

has kernel $\{\alpha(1,1)\}$, so the eigenspace $E_6 = \text{span}\{(1,1)\}$.

$$A - 2I = \begin{bmatrix} 3 & 1 \\ 3 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

has kernel $\{\alpha(-1,3)\}$, so the eigenspace $E_2 = \text{span}\{(-1,3)\}$.

Example 5.36. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$.

The characteristic equation is

$$\begin{aligned} 0 = \chi_A(\lambda) &= \left| \begin{pmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{pmatrix} \right| \\ &= (2-\lambda)(-2-\lambda)(2-\lambda) - 3 - 3 - ((-2-\lambda) - 3(2-\lambda) - 3(2-\lambda)) \\ &= -\lambda^3 + 2\lambda^2 + 4\lambda - 8 - 6 + 2 + \lambda + 12 - 6\lambda \\ &= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda-1)^2 \end{aligned}$$

so the eigenvalues are 0 and 1 (twice). We have

$$A - 0I = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\ker(A) = \{\alpha(1,1,1)\}$, and $E_0 = \text{span}\{(1,1,1)\}$. We also have

$$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\ker(A - I) = \{\alpha(3,1,0) + \beta(-1,0,1)\}$, and $E_1 = \text{span}\{(3,1,0), (-1,0,1)\}$.

Proposition 5.37. *If A is a $n \times n$ matrix and n is odd, then A has at least one eigenvalue.*

Proof. Recall that a degree n polynomial always has at least one real root if n is odd. Thus if $A \in M_{n\times n}$, $\chi_A(\lambda)$ is degree n, and has a real root, which is an eigenvalue of A. □

Example 5.38. Find the eigenvalues and corresponding eigenspaces of $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Since this matrix is triangluar, we know the eigenvalues are $2, 4, 2$. We solve

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\ker(A - 2I) = \{\alpha(0, 0, 1)\}$, so $E_2 = \operatorname{span}\{(0, 0, 1)\}$. Similarly,

$$A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\ker(A - 4I) = \{\alpha(0, 1, 0)\}$ so $E_4 = \operatorname{span}\{(0, 1, 0)\}$.

Notice that in this case, the span of the eigenvectors is only 2-dimensional; the eigenvectors don't span the whole domain.

6 Similarity and Change of Basis

6.1 Change of Basis

In the last section we said that two vector spaces are isomorphic if they're essentially the same, at least from the perspective of vector spaces. (Polynomials are not the same thing as lists of real numbers. But they are the same as far as the vector space structure goes). From this we should immediately assume that every space is isomorphic to *itself.*

Proposition 6.1. *Let V be a vector space. Then $V \cong V$. In particular, the identity map Id_V defined by $Id_V(\mathbf{v}) = \mathbf{v}$ is an isomorphism from V to V.*

But two isomorphic vector spaces can have more than one isomorphism between them. In fact, any non-trivial vector space has infinitely many isomorphisms from itself to itself, and these isomorphisms are extremely useful.

Proposition 6.2. *Let U, V be vector spaces, let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for U, and $L : U \to V$ be a linear map. Then $L(E)$ spans $L(U)$, and L is an isomorphism if and only if $L(E)$ is a basis for V.*

Corollary 6.3. *Let U, V be vector spaces, with* $\dim U = \dim V$. *Then $U \cong V$.*

Proof. Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for U, and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ a basis for V; we know they have the same number of elements since $\dim U = \dim V$. Define a linear map $L : U \to V$ by linearly extending $L(\mathbf{e}_i) = \mathbf{f}_i$; that is, we define

$$L(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = a_1\mathbf{f}_1 + \cdots + a_n\mathbf{f}_n.$$

This map sends a basis to a basis, and thus is an isomorphism. □

Corollary 6.4. *A linear map from V to V is an isomorphism if and only if it sends a basis to a basis.*

Definition 6.5. We call such an isomorphism a *change of basis map.* The matrix of such an isomorphism is called *transition matrix.*

Example 6.6. We know that $E = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

are both bases for $\mathbb{R}^3$. Define $L : \mathbb{R}^3 \to \mathbb{R}^3$ by setting

$$\begin{aligned} L(1,0,0) &= (1,0,0) \\ L(0,1,0) &= (1,1,0) \\ L(0,0,1) &= (1,1,1) \\ A &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Then L is an isomorphism and a change of basis from E to F.

We can find the inverse in one of two ways. One is to use row reduction as usual, but that takes effort. Instead, we can note that the inverse is just the map that sends F to E:

$$\begin{aligned} L^{-1}(1,0,0) &= (1,0,0) \\ L^{-1}(1,1,0) &= (0,1,0) = (1,1,0) - (1,0,0) \\ L^{-1}(1,1,1) &= (0,0,1) = (1,1,1) - (1,1,0) \\ A^{-1} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

By multiplication or substitution we can check that this is definitely an inverse.

Now let's ask a separate question. Suppose we have a vector $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ that is currently expressed in terms of the standard basis, and we would like to find its coordinates in F. This means that we want to write it as a sum of things in F, and so we want to solve the equation

$$x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

If we write this equation in matrix form, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Notice that the matrix in this equation is just the matrix A we found that sends elements of E to elements of F.

We could solve this system of equations by row-reducing. But we can also just compute the inverse matrix: if $A\mathbf{x} = \mathbf{u}$ then $\mathbf{x} = A^{-1}\mathbf{u}$. And as we saw, the matrix A^{-1} is just the matrix that sends elements of F to elements of E. In this context, we call A the transition matrix from F to E, and A^{-1} the transition matrix from E to F.

Example 6.7 (continued)**.** Suppose we'd like to take the vector $\mathbf{u} = 2(1,0,0) + 3(1,1,0) + 5(1,1,1)$ and find its coordinates in the standard basis. We have

$$\begin{aligned}
\mathbf{u} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\\0\end{bmatrix} + 5\begin{bmatrix}1\\1\\1\end{bmatrix} &= 2(\mathbf{e}_1) + 3(\mathbf{e}_1 + \mathbf{e}_2) + 5(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\
&= (2+3+5)\mathbf{e}_1 + (3+5)\mathbf{e}_2 + 5\mathbf{e}_3 \qquad \text{or} \\
[\mathbf{u}]_E = A[\mathbf{u}]_F = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\3\\5\end{bmatrix} &= \begin{bmatrix}10\\8\\5\end{bmatrix}.
\end{aligned}$$

In many ways it's more useful to do things the other way. Suppose we have the vector $(5,2,7)$ and want to express it as a linear combination of elements of F. Then we need the transition matrix from F to E, which is A^{-1}. So we have

$$A^{-1}\begin{bmatrix}5\\2\\7\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\2\\7\end{bmatrix} = \begin{bmatrix}3\\-5\\7\end{bmatrix}.$$

Thus

$$\begin{bmatrix}5\\2\\7\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} - 5\begin{bmatrix}1\\1\\0\end{bmatrix} + 7\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Example 6.8. Let's represent the polynomial $a + bx + cx^2 \in \mathcal{P}_3(x)$ as a linear combination of $F = \{1, 2x, 4x^2 - 2\}$.

We take $E = \{1, x, x^2\}$ to be the standard basis, and if A is the transition matrix from

F to E we have

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}.$$

Thus we have

$$[a + bx + cx^2]_F = A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + c/2 \\ b/2 \\ c/4 \end{bmatrix}.$$

Thus $[a + bx + cx^2]_F = (a + c/2, b/2, c/4)$ and

$$a + bx + cx^2 = (a + c/2)(1) + (b/2)(2x) + (c/4)(4x^2 - 2).$$

I want to mention one last idea here, which is the ability to paste transition matrices together. If A is the transition matrix from F to E, and B is the transition matrix from G to F, then AB is the transition matrix from G to E. This is primarily useful once we introduce inverses.

Example 6.9. Let $E = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}, F = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^3$. Let $\mathbf{u} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Let's find the coordinates of $\mathbf{u}$ with respect to F.

We could try to compute the transition matrix directly, but that requires us to do a bunch of equation solving. Instead, we notice that the transition matrix from E to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

We want the transition matrix from E to F so that we can convert coordinates from E to F. Thus the matrix we actually want is $B^{-1}A$: A takes us from E to the standard basis, and then B^{-1} takes us from the standard basis to F. We compute

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right]$$

and thus

$$B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B^{-1}A = \frac{1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}$$

$$[\mathbf{u}]_F = B^{-1}A[\mathbf{u}]_E = \frac{1}{2}\begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 7/2 \\ 7/2 \end{bmatrix}.$$

We check that, indeed,

$$\frac{-3}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{7}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{7}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

6.2 Similarity

We now want to return to talking about general linear transformations, but bringing with us our new perspective on bases and changes of bases.

Let $R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by a rotation ninety degrees counterclockwise. We saw earlier that with respect to the standard basis, this transformation has matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. But we can also compute the matrix with respect to, say, $F = \{(1,0),(1,1)\}$. Then we have

$$\begin{aligned} R_{\pi/2}(1,0) &= (0,1) = (1,1) - (1,0) \to (-1,1) \\ R_{\pi/2}(1,1) &= (-1,1) = (1,1) - 2(1,0) \to (-2,1) \\ B &= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

These two matrices represent the same transformation, with respect to different bases. But they are clearly not the same matrix! What's going on here?

The answer is that we changed the coordinate system, and so our matrix changed. After we account for that, we should get the same matrix. To account for this, we need the change of basis matrix between F and the standard basis E. We have

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

the transition matrix from F to the standard basis, and thus

$$U^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

is the transition matrix from the standard basis to F.

If we want to perform the operation $R_{\pi/2}$ on the vectors of F, we can use the matrix B that we found. Alternatively, we can transform our vectors into E-coordinates, use the matrix A, and then transform back into F-coordinates. This operation would be given by $U^{-1}AU$. We calculate that

$$U^{-1}AU = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

This is the same as the matrix B, as it should be.

Definition 6.10. If A and B are $n \times n$ matrices, we say they are *similar* if there is some invertible matrix U such that $B = U^{-1}AU$. We write $A \sim B$.

Proposition 6.11. *Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}, F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases for V, and let $L : V \to V$ be a linear function. Let U be the transition matrix from F to E.*

If A is the matrix representing L with respect to E, and B is the matrix representing L with respect to F, then $B = U^{-1}AU$.

Example 6.12. Let $D : \mathcal{P}_2(x) \to \mathcal{P}_2(x)$ be the differentiation operator. Let's find the matrix of D with respect to $E = \{1, x, x^2\}$ and with respect to $F = \{1, 2x, 4x^2 - 2\}$.

We've already seen that the matrix of D with spect to E is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

We can work out the matrix with respect to F directly:

$$\begin{aligned} D(1) &= 0 \to (0,0,0) \\ D(2x) &= 2 \to (2,0,0) \\ D(4x^2-2) &= 8x \to (0,4,0) \\ B &= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Alternatively, we could recall that the change of basis matrices between E and F:

$$\begin{aligned} E \to F : \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} &= U^{-1} \\ F \to E : \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} &= U. \end{aligned}$$

So we can compute the matrix B for D by saying

$$\begin{aligned} B = U^{-1}AU &= \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Example 6.13. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $L(x, y, z) = (x + 3y + z, 2x - y + 3z, y - z)$. Find the matrix of L with respect to $\{(4, 1, 2), (3, 0, 1), (1, -1, 0)\}$, and show it is similar to the matrix with respect to the standard basis.

We have

$$\begin{aligned}
L(1,0,0) &= (1,2,0)\\
L(0,1,0) &= (3,-1,1)\\
L(0,0,1) &= (1,3,-1)\\
A &= \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix}.
\end{aligned}$$

We can compute the change of basis matrices. If U is the matrix from F to E, then we have

$$U = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

and

$$\left[\begin{array}{ccc|ccc} 4 & 3 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 4 & 3 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 1 & -4 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 3 & 5 & 1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & -1 & 1 & 2 & -3 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & -2 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 3 \\ 0 & 1 & 0 & 2 & 2 & -5 \\ 0 & 0 & 1 & -1 & -2 & 3 \end{array}\right]$$

so we have

$$U^{-1} = \begin{bmatrix} -1 & -1 & 3 \\ 2 & 2 & -5 \\ -1 & -2 & 3 \end{bmatrix}.$$

Thus to find the matrix with respect to F, we can compute

$$B = U^{-1}AU = \begin{bmatrix} -1 & -1 & 3 \\ 2 & 2 & -5 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 & -7 \\ 6 & -1 & 13 \\ -5 & 2 & -10 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -25 & -16 & -4 \\ 49 & 31 & 7 \\ -38 & -25 & -7 \end{bmatrix}.$$

There's not a really efficient way to determine whether two matrices are similar in general, although we have a few tools that can tell us two matrices are *not* similar.

Proposition 6.14. *Let $A, B \in M_{n\times n}$ with $A \sim B$. Then:*

- *The rank of A is equal to the rank of B.*
- *The nullity of A is equal to the nullity of B.*
- *A is invertible if and only if B is invertible.*

6.3 Determinant and Trace

Because matrices that are similar represent the same underlying transformation, anything that's a property of the transformation, and not of the particular matrix, should remain unchanged. Thus two matrices representing the same linear transformation should have the same eigenvalues. And indeed, this is the case.

Proposition 6.15. *Suppose A and B are similar $n \times n$ matrices, so there exists U such that $B = U^{-1}AU$. Then:*

- $\det(A) = \det(B)$
- $\chi_A(\lambda) = \chi_B(\lambda)$
- *A and B have the same set of eigenvalues.*

Proof. We can prove these two ways. From a formal perspective, we know that A and B must represent the same linear transformation, and since all of these things are properties of the linear transformation, they must be the same for similar matrices.

From a more concrete algebraic perspective, we have:

- $\det(B) = \det(U^{-1}AU) = \det(U^{-1})\det(A)\det(U) = \frac{1}{\det(U)}\det(A)\det(U) = \det(A)$.
- For any λ, we have

  $$U^{-1}(A-\lambda I)U = U^{-1}AU - U^{-1}\lambda IU = B - \lambda I,$$

  so we have $(A-\lambda I) \sim (B - \lambda I)$. By the previous result, $\det(A-\lambda I) = \det(B-\lambda I)$.
- The eigenvalues are the roots of the characteristic polynomial. Since $\chi_A(\lambda) = \chi_B(\lambda)$, the roots are the same and so the eigenvalues are the same.

$\square$

Example 6.16. Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and let $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, so that

$$B = U^{-1}AU = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}.$$

Clearly $A \sim B$. We can see immediately that $\chi_A(\lambda) = (2-\lambda)(1-\lambda)(1-\lambda) = 2 - 5\lambda + 4\lambda^2 - \lambda^3$ and $\det(A) = 2$. With a little more work, we have

$$\begin{aligned} \chi_B(\lambda) = \det \begin{bmatrix} -\lambda & 2 & 0 \\ 2 & 3-\lambda & 3 \\ 1 & -2 & 1-\lambda \end{bmatrix} \\ &= -\lambda\left((3-\lambda)(1-\lambda) - (-2\cdot 3)\right) - 2\left(2(1-\lambda) - 1\cdot 3\right) \\ &= -3\lambda + \lambda^2 + 3\lambda^2 - \lambda^3 - 6\lambda - 4 + 4\lambda + 6 \\ &= 2 - 5\lambda + 4\lambda^2 - \lambda^3 = \chi_A(\lambda). \end{aligned}$$

Remark 6.17. The converse of this theorem is not true. Similar matrices always have the same characteristic polynomial; but sometimes matrices with the same characteristic polynomial are not similar.

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then $\chi_A(\lambda) = (1-\lambda)^2 = \chi_I(\lambda)$. But clearly A is not similar to the identity, since $U^{-1}IU = I$, and so the only matrix similar to I is itself.

Since the characteristic polynomials of similar matrices are the same, they clearly have all the same coefficients. In fact, we can see that the determinant is just the constant term of the characteristic polynomial, $\chi_A(0)$. There's one other coefficient that's often important. It's not the highest degree coefficient, which is always ± 1; but the second-highest coefficient is often interesting and useful.

Definition 6.18. If $L : V \to V$ is a linear transformation on a n-dimensional vector space, we define the *trace* of L to be $\operatorname{Tr}(L) = (-1)^{n-1} a_{n-1}$ where $\chi_L(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$.

If A is a $n \times n$ matrix, we define the *trace* of A to be the trace of the linear transformation represented by A. Thus $\operatorname{Tr}(A) = (-1)^{n-1} a_{n-1}$ where $\chi_A(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$.

Proposition 6.19.

- *If $A \sim B$ then $\operatorname{Tr}(A) = \operatorname{Tr}(B)$.*
- *$\operatorname{Tr}(A)$ is the sum of the eigenvalues of A (weighted by multiplicity).*
- *$\operatorname{Tr}(A) = \sum_{i=1}^n a_{ii}$ is the sum of the entries on the main diagonal of A.*

Proof.

- This follows from the fact that $\chi_A(\lambda) = \chi_B(\lambda)$.
- If $\chi_A(\lambda) = (\lambda - \lambda_1)^{d_1} \dots (\lambda - \lambda_k)^{d_k}$, then when we multiply this out, the $n-1$ coefficient will be $\sum_{i=1}^k d_k \lambda_k$.
- Proof by induction.

□

Remark 6.20. This tells us that the trace is very easy to compute; unlike the determinant, it doesn't depend on any non-diagonal entries, and just requires some fast, simple addition.

This also tells us that the trace is a *similarity invariant*, meaning that similar matrices have the same trace. Thus we can quickly test whether two matrices might be similar by computing the traces of both.

But notice that, like with the determinant, we can have two matrices which are not similar but have the same trace.

If the matrix A is given as a function of T, there is a specific sense in which the trace is related to the derivative of the determinant.

Proposition 6.21. *The trace is a linear multiplicative map on matrices. That is:*

- $\operatorname{Tr}(A + B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$
- $\operatorname{Tr}(rA) = r \operatorname{Tr}(A)$
- $\operatorname{Tr}(A^T) = \operatorname{Tr}(A)$

Proof. These follow from the characterization of the trace as the sum of the diagonal elements. □

Example 6.22. Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 1 \\ 2 & -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 & 1 \\ 4 & -3 & -3 \\ 2 & 1 & 0 \end{bmatrix}$. Then $\operatorname{Tr}(A) = 3+4+2 = 9$ and $\operatorname{Tr}(B) = 5-3+0 = 2$, so we know that $A \not\sim B$.

If $C = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 1 & 7 \\ 1 & 1 & 4 \end{bmatrix}$ then $\operatorname{Tr}(C) = 4+1+4 = 9$, so it's possible that $C \sim A$. But we'd need to do more work to confirm this. On just this evidence, it probably isn't.

In fact we can compute that $\chi_A(\lambda) = -x^3+9x^2-24x+26 \neq -x^3+9x^2-17x-22 = \chi_C(\lambda)$, so the matrices actually aren't similar.

6.4 Diagonalization

We now reach the payoff to all this discussion of changes of coordinates. If we find the eigenvectors of a linear operator and the give us a (eigen)basis for our space, we can always find a matrix representation of our linear operator with a particularly *nice* matrix by working with respect to this eigenbasis. In particular:

Definition 6.23. If D is a $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \neq j$, we say that D is *diagonal*.

Proposition 6.24. *Let* $D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}$ *be a diagonal* $n \times n$ *matrix. Then:*

- *Each standard basis vector* $\mathbf{e}_i$ *is an eigenvector of* D *with eigenvalue* d_{ii}.
- $\det(D) = \prod_{i=1}^n d_{ii}$ *is the product of the diagonal entries.*
- $\mathbb{R}^n$ *is spanned by the eigenvectors of* D.

Proof.
- We have

$$D\mathbf{e}_i = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ d_{ii} \\ \vdots \\ 0 \end{bmatrix} = d_{ii}\mathbf{e}_i.$$

- The determinant is the product of the eigenvalues, which are the diagonal entries.
- The standard basis vectors are eigenvalues, and span $\mathbb{R}^n$.

□

Definition 6.25. We say a linear transformation is *diagonalizable* if its matrix in some basis is diagonal.

We say a matrix is *diagonalizable* if its linear transformation is diagonalizable. Thus A is diagonalizable if A is similar to some diagonal matrix.

Proposition 6.26. *Let A be a $n \times n$ matrix. Then:*

1. *A is diagonalizable if and only if the eigenvectors of A span $\mathbb{R}^n$.*
2. *A is diagonalizable if and only if it has n linearly independent eigenvectors.*
3. *If A has n distinct eigenvalues, then A is diagonalizable.*

Proof.
1. Suppose A is diagonalizable, i.e. there is an invertible matrix U and a diagonal matrix D such that $A = U^{-1}DU$. Let F be the image of the standard basis under U^{-1}; then

$$A\mathbf{f}_i = U^{-1}DU\mathbf{f}_i = U^{-1}D\mathbf{e}_i = U^{-1}d_{ii}\mathbf{e}_i = d_{ii}U^{-1}\mathbf{e}_i = d_{ii}\mathbf{f}_i.$$

Thus $\mathbf{f}_i$ is an eigenvector for each i, so we have a basis of eigenvectors.

Conversely Suppose the eigenvectors of A span $\mathbb{R}^n$. Then in particular there is a basis $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ of eigenvectors. Let U be the matrix that sends the standard basis to F. Then for each i we have

$$U^{-1}AU\mathbf{e}_i = U^{-1}A\mathbf{f}_i = U^{-1}\lambda_i\mathbf{f}_i = \lambda_i U^{-1}\mathbf{f}_i = \lambda_i\mathbf{e}_i$$

and thus $U^{-1}AU$ is a diagonal matrix with $d_{ii} = \lambda_i$. Thus A is diagonalizable.

2. A set of n linearly independent vectors is a basis for $\mathbb{R}^n$. Thus A has n linearly indpendent eigenvectors if and only if the eigenvectors span $\mathbb{R}^n$.

3. Let $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ be a set of eigenvectors corresponding to each eigenvalue. Then this set is linearly independent by proposition 5.8, and thus A has n linearly independent eigenvectors.

□

Remark 6.27. Notice that the converse of (3) is not true. For instance, the identity has only one eigenvalue, but is clearly diagonalizable (and already diagonalized).

Corollary 6.28. *If A is a $n \times n$ matrix and $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is a basis of eigenvectors, and U is the matrix sending the standard basis to F, then $D = U^{-1}AU$ is a diagonal matrix.*

We say that the matrix U diagonalizes *A.*

Remark 6.29. Diagonalization is not unique; the matrix U depends on the choice of basis. However, since the diagonal entries are the eigenvalues, they will be the same (up to reordering) for any diagonalization.

Example 6.30. Let $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$. We know that the eigenvalues are 4 and -3, so the matrix is diagonalizable; the corresponding eigenvectors are $(2, 1)$ and $(-1, 3)$. So we set

$$\begin{aligned} U &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \\ U^{-1} &= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \\ U^{-1}AU &= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 4 & -9 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 28 & 0 \\ 0 & -21 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}. \end{aligned}$$

Example 6.31. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. We saw that the eigenvalues are $0, 1, 1$. The eigenvectors are $(1,1,1),(3,1,0),(-1,0,1)$, so we set

$$U = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$U^{-1}AU = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 6.32. We saw that the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ had eigenspaces $E_2 = \operatorname{span}\{(0,0,1)\}$ and $E_4 = \operatorname{span}\{(0,1,0)\}$. The eigenvectors do not span $\mathbb{R}^3$, so A is not diagonalizable.

In general I don't really expect triangular matrices with repeated eigenvalues to be diagonal, but treating this thought fully is beyond the scope of this course.

There are two different major uses for diagonalization. The first is to tell us the basis we "should" be working in, and to allow us to change bases to that basis. The basis in which your operator is diagonal is the basis in which your operator is "really" working; it divides your space up into the dimensions along which your operator really works.

Eigenvectors and diagonalization are often used in various sorts of data analysis. The eigenvector corresponding to the largest eigenvalue is the most significant input, so diagonalization can tell us which components of our data are most important to whatever phenomenon we're studying; this is the idea behind "principal component analysis".

They are also used in various sorts of approximate computations: if your linear operator has eigenvalues of 5, 3, 1, .1, .1, -.1, .0005, you can get a pretty good approximation of your operator by ignoring the eigenvectors corresponding to the small eigenvalues, and only worrying about the large ones.

Second, we can use diagonalization to simplify many matrix computations. We need to make two observations: one about diagonal matrices, the other about similar matrices.

Proposition 6.33. *Suppose C and D are two diagonal matrices with diagonal entries given by c_{ii}, d_{ii} respectively. Then their product is a diagonal matrix given by*

$$\begin{bmatrix} c_{11} & 0 & 0 & \dots & 0 \\ 0 & c_{22} & 0 & \dots & 0 \\ 0 & 0 & c_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn} \end{bmatrix} = \begin{bmatrix} c_{11}d_{11} & 0 & 0 & \dots & 0 \\ 0 & c_{22}d_{22} & 0 & \dots & 0 \\ 0 & 0 & c_{33}d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{nn}d_{nn} \end{bmatrix}.$$

Proposition 6.34. *If $A = U^{-1}BU$, then $A^n = U^{-1}B^nU$.*

Proof.

$$\begin{aligned} A^n = (U^{-1}BU)^n &= U^{-1}BUU^{-1}BU \dots U^{-1}BUU^{-1}BU \\ U^{-1}BI_nB \dots IBIBU &= U^{-1}BB \dots BBU = U^{-1}B^nU. \end{aligned}$$

□

Example 6.35. Let Let $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$. Find A^5.

If $U^{-1}AU = D$, then $UU^{-1}AUU^{-1} = UDU^{-1}$ and thus $A = UDU^{-1}$. So

$$\begin{aligned} D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} &= \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = U^{-1}AU \\ A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = UDU^{-1} \\ A^5 = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}^5 &= \left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right)^5 \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}^5\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1024 & 0 \\ 0 & -243 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3072 & 1024 \\ 243 & -486 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 5901 & 2534 \\ 3801 & -434 \end{bmatrix} = \begin{bmatrix} 843 & 362 \\ 543 & -62 \end{bmatrix}. \end{aligned}$$

Example 6.36. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. Find a formula for A^n.

We have

$$\begin{aligned}
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
\begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}^n &= \left(\begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \right)^n \\
&= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.
\end{aligned}$$

Corollary 6.37. *If A is a diagonalizable matrix whose eigenvalues are only zero or one, then $A^n = A$ for any n.*

Markov chains blah

Example 6.38. $B = \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}$ has eigenvalues $1/2, 1$ with eigenvectors $(1, -1)$ and $(2, 3)$.

$$\begin{aligned}
U &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\
U^{-1} &= \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \\
U^{-1}BU &= \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .5 & 2 \\ -.5 & 3 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 2.5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \\
B = UDU^{-1} &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \\
B^n &= \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \right)^n \\
&= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix}^n \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \\
\lim_{n\to\infty} B^n &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}.
\end{aligned}$$

7 Inner Product Spaces and Geometry

In this section we're going to consider vector spaces from a more geometric perspective. In $\mathbb{R}^3$ we have the geometric ideas of "distance" and "angle", but neither of those is necessarily present in an arbitrary vector space. Here we will introduce a new structure called an "Inner Product" that allows us to generalize the angles and distances of $\mathbb{R}^3$ to any vector space with an inner product structure.

7.1 The Dot Product

Definition 7.1. Let $\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$. We define the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n = \sum_{i=1}^n u_i v_i.$$

This is sometimes also called the *scalar product* on $\mathbb{R}^n$.

Remark 7.2. If we think of $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices, we can think of $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$, the product of a $n \times 1$ matrix with a $1 \times n$ matrix.

The dot product has a number of useful properties. First of all, it allows us to define the length or magnitude of a vector.

Definition 7.3. Let $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$. We define the *magnitude* of $\mathbf{v}$ to be

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Notice that this is just the usual definition of distance; in the plane this is

$$\|(x, y)\| = \sqrt{x^2 + y^2},$$

which is just the pythagorean theorem.

Sometimes it's useful to talk about the distance between two points, rather than the length of a vector. But the distance between two points is the length of the vector between them, so we can define the distance between $\mathbf{x}$ and $\mathbf{y}$ to be

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

The dot product has a few important properties:

Proposition 7.4. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then:*

1. *(Positive definite)* $\mathbf{u} \cdot \mathbf{u} \geq 0$*, and if* $\mathbf{u} \cdot \mathbf{u} = 0$ *then* $\mathbf{u} = \mathbf{0}$.
2. *(Symmetric)* $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
3. *(Bilinear) The function defined by* $L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$ *is linear, and the function defined by* $T(\mathbf{y}) = \mathbf{u} \cdot \mathbf{y}$ *is linear.*

Proof.

1. $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots u_n^2$. Each term is non-negative since each term is a real square, so the sum is non-negative. The sum is zero if and only if each term is zero, if and only if $\mathbf{u} = (0, \dots, 0) = \mathbf{0}$.
2. $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n = v_1 u_1 + \cdots + v_n u_n = \mathbf{v} \cdot \mathbf{u}$.
3. We'll prove linearity in the first coordinate; the proof for the second coordinate is identical.

   Fix $\mathbf{v} \in \mathbb{R}^n$ and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Define $L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$. Then

$$\begin{aligned} L(r\mathbf{x}) &= (r\mathbf{x}) \cdot \mathbf{v} = (rx_1)v_1 + \cdots + (rx_n)v_n = r(x_1 v_1 + \cdots + x_n v_n) = rL(\mathbf{x}) \\ L(\mathbf{x} + \mathbf{y}) &= (\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = (x_1 + y_1)v_1 + \cdots + (x_n + y_n)v_n \\ &= (x_1 v_1 + \cdots + x_n v_n) + (y_1 v_1 + \cdots + y_n v_n) = L(\mathbf{x}) + L(\mathbf{y}). \end{aligned}$$

□

The dot product also allows us to compute the angle between two vectors.

Proposition 7.5. *If* $\mathbf{u}, \mathbf{v}$ *are two nonzero vectors in* $\mathbb{R}^n$*, and the angle between them is* θ*, then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta.$$

Proof. We can form a triangle with sides $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$. Then by the law of cosines (which I'm sure you all remember from high school trigonometry), we have

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta.$$

Then we compute

$$\begin{aligned} \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta &= \frac{1}{2}\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - \|\mathbf{y} - \mathbf{x}\|^2\right) \\ &= \frac{1}{2}\left(\mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})\right) \\ &= \frac{1}{2}\left(\mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} - (\mathbf{y} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{x})\right) \\ &= \frac{1}{2}\left(\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x}\right) = \mathbf{x} \cdot \mathbf{y}. \end{aligned}$$

□

Thus the angle between two vectors is given by $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

Example 7.6. Let $\mathbf{u} = (3,4)$ and $\mathbf{v} = (-1,7)$. Then $\mathbf{u}\cdot\mathbf{v} = 3\cdot(-1) + 4\cdot 7 = 25$.

We can compute $\|u\| = \sqrt{3^2+4^2} = 25$ and $\|\mathbf{v}\| = \sqrt{(-1)^2+7^2} = 5\sqrt{2}$. The distance between them is $\|\mathbf{u}-\mathbf{v}\| = \|(4,-3)\| = \sqrt{4^2+(-3)^2} = 5$.

The angle between them is given by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{25}{5\cdot 5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

We sometimes want to be able to talk about the direction of a vector without worrying about the magnitude. In this case we may wish to comput the *unit vector* given by $\frac{\mathbf{u}}{\|\mathbf{u}\|}$. This vector will clearly have magnitude 1, and point in the same direction that $\mathbf{u}$ does.

If $\mathbf{x},\mathbf{y}$ are unit vectors, then $\cos\theta = \mathbf{x}\cdot\mathbf{y}$.

Example 7.7. The unit vector of $\mathbf{u} = (3,4)$ is $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{5}(3,4) = (3/5, 4/5)$. The unit vector of $\mathbf{v} = (-1,7)$ is $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5\sqrt{2}}(-1,7) = \left(\frac{-1}{5\sqrt{2}}, \frac{7}{5\sqrt{2}}\right)$.

Then the angle between them is given by

$$\cos\theta = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \cdot \begin{bmatrix} -1/5\sqrt{2} \\ 7/5\sqrt{2} \end{bmatrix} = \frac{-3}{25\sqrt{2}} + \frac{28}{25\sqrt{2}} = \frac{1}{\sqrt{2}}$$

as before.

There is one more result that is pretty trivial in the case of $\mathbb{R}^n$, but will be very important when we generalize.

Theorem 7.8 (Cauchy-Schwarz Inequality)**.** *If* $\mathbf{u},\mathbf{v}$ *are vectors in* $\mathbb{R}^n$*, then*

$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|. \tag{2}$$

Furthermore, the two sides are equal if and only if either one of the vectors is $\mathbf{0}$*, or* $\mathbf{u} = r\mathbf{v}$ *for some* $r\in\mathbb{R}$*.*

Proof. Recall that $0 \leq |\cos\theta| \leq 1$, with $|\cos\theta| = 1$ if and only if $\theta = n\pi$ for some integer n. Thus

$$|\mathbf{u}\cdot\mathbf{v}| = |\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Further, the equality holds only if $\|\mathbf{u}\| = 0$, $\|\mathbf{v}\| = 0$, or $\cos\theta = 1$. In the third case this means the angle between the two vectors is an integer multiple of π, so they either point in the same direction, or in opposite directions. □

180 degree angles are important, but so are right angles. If two vectors are at a right angle to each other, then we have

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \pi/2 = \|\mathbf{u}\| \|\mathbf{v}\| \cdot 0 = 0.$$

We give a special name to these vectors:

Definition 7.9. We say that $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 7.10. 1. $\mathbf{0}$ is orthogonal to every vector.

2. $(3, 2)$ and $(-4, 6)$ are orthogonal in $\mathbb{R}^2$.

3. Let $\mathbf{u} = (2, 3, 2)$. Can we find a vector orthogonal to it?

   There are lots of them. (They should form an entire plane, if you think about it). One in particular is $(1, 1, -5/2)$.

The last important idea the dot product gives us is the ability to break a vector up into two components. Given $\mathbf{u}$ and $\mathbf{v}$, we can decompose $\mathbf{u}$ into "the part that points in the direction of $\mathbf{v}$" and "the other part."

Suppose we have two vectors $\mathbf{u}$ and $\mathbf{v}$, with angle θ between them. These form two sides of a triangle, with the third side given by $\mathbf{u} - \mathbf{v}$. But we can also draw a line from the endpoint of $\mathbf{u}$ that is perpendicular to $\mathbf{v}$.

We now have a right trangle. The hypotenuse has length $\|\mathbf{u}\|$, so by definition of cosine the length of the adjacent side is $\|\mathbf{u}\| \cos \theta$. But we know that

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\ \mathbf{u} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} &= \|\mathbf{u}\| \cos \theta \end{aligned}$$

so the length of the adjacent side is $\mathbf{u} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$. We sometimes call this number the *scalar projection of* $\mathbf{u}$ *onto* $\mathbf{v}$.

Further, we know the direction that the adjacent side is pointing: it's the same direction as $\mathbf{v}$! So we can find this adjacent side as a vector with the formula

$$\mathbf{p} = \mathbf{u} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

It is not immediately obvious that this is a vector; but most of the dot products give us scalars, with the final $\mathbf{v}$ giving direction.

Finally, we can write $\mathbf{w} = \mathbf{u} - \mathbf{p}$. We will have that $\mathbf{p} \cdot \mathbf{v} = \|\mathbf{p}\|\|\mathbf{v}\|$ since the two vectors point in the same direction; we will have

$$\begin{aligned}
\mathbf{w} \cdot \mathbf{v} &= (\mathbf{u} - \mathbf{p}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \mathbf{p} \cdot \mathbf{v} \\
&= \mathbf{u} \cdot \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}(\mathbf{v} \cdot \mathbf{v}) \\
&= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} = \mathbf{0}.
\end{aligned}$$

Thus $\mathbf{w}$ is orthogonal to $\mathbf{v}$. We have written $\mathbf{u} = \mathbf{p} + \mathbf{w}$ so that $\mathbf{w}$ is orthogonal to $\mathbf{v}$, and $\mathbf{p}$ points in the same direction as $\mathbf{v}$.

Definition 7.11. If $\mathbf{u}, \mathbf{v}$ are two vectors in $\mathbb{R}^n$, we define the *projection map onto* $\mathbf{v}$ by

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}.$$

Example 7.12. Let's look back at our earlier vectors $\mathbf{u} = (3, 4)$ and $\mathbf{v} = (-1, 7)$. Then we compute

$$\begin{aligned}
\mathrm{proj}_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{25}{50}\begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix} \\
\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\mathbf{u} &= \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 1/2 \end{bmatrix}.
\end{aligned}$$

7.2 Inner Products

All the ideas of the previous section work in $\mathbb{R}^n$. We want to figure out what the important bits were so that we can use them in other vector spaces. Clearly the most important part was the dot product.

Definition 7.13. An *inner product* on a vector space V is an operation that takes in two vectors $\mathbf{u}, \mathbf{v} \in V$ and returns a real number $\langle \mathbf{u}, \mathbf{v} \rangle$, satisfying the following conditions:

1. (Positive Definite) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
2. (Symmetric) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
3. (Bilinear) $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$, and $\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{u}, \mathbf{w} \rangle$.

We write the *norm* of a vector $\mathbf{v}$ as $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

The dot product is clearly an example of an inner product, but there are other important examples we can see.

Example 7.14. Let $V = \mathcal{C}([a,b],\mathbb{R})$ be the space of continuous functions on $[a,b]$, and define an inner product by $\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$. Then

1. $\langle f, f\rangle = \int_a^b f(t)^2\,dt \geq 0$ since $f(t)^2 \geq 0$; and further the integral is zero if and only if $f(t)^2 = 0$ everywhere.
2. $\langle f, g\rangle = \int_a^b f(t)g(t)\,dt = \int_a^b g(t)f(t)\,dt = \langle g, f\rangle$.
3. $\langle \alpha f + \beta g, h\rangle = \int_a^b(\alpha f(t) + \beta g(t))h(t)\,dt = \alpha\int_a^b f(t)h(t)\,dt + \beta\int_a^b g(t)h(t)\,dt = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$.

Thus this is an inner product on $\mathcal{C}([a,b],\mathbb{R})$ by definition.

Example 7.15. Let $V = \mathcal{P}_n(x)$ and fix real numbers $x_0, x_1, \dots, x_n$ be distinct real numbers. For $f, g \in V$, define

$$\langle f, g\rangle = \sum_{i=0}^n f(x_i)g(x_i).$$

Then we can see $\langle f, f\rangle = \sum_{i=0}^n f(x_i)^2 \geq 0$, and the sum is equal to zero if and only if $f(x_i) = 0$ for all i. But then f is a degree n polynomial with $n+1$ roots, and so must be constantly zero.

You will check the other two conditions on your homework.

We'd like to check that this inner product gives us all the things that the dot product did. In particular we want it to give us distance and angle and projections.

Definition 7.16. Let $\mathbf{u}, \mathbf{v}$ be elements of an inner product space V. If $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, we say that $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal*.

We will eventually see that this corresponds to the two vectors being at a "right angle" to each other. But more immediately, we'll see that this means they are independent in a very specific way.

Definition 7.17. Suppose $\mathbf{u}, \mathbf{v}$ are vectors in an inner product space V, and $\mathbf{v} \neq 0$. We define the projection of $\mathbf{u}$ onto $\mathbf{v}$ by

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}.$$

Proposition 7.18. *Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space V, with $\mathbf{v} \neq 0$. Let $\mathbf{p} = \operatorname{proj}_{\mathbf{v}}\mathbf{u}$. Then:*

1. $\langle \mathbf{u} - \mathbf{p}, \mathbf{p}\rangle = 0$*—that is,* $\mathbf{u} - \mathbf{p}$ *is orthogonal to* $\mathbf{p}$.

2. $\mathbf{u} = \mathbf{p}$ *if and only if* $\mathbf{u}$ *is a scalar multiple of* $\mathbf{v}$*.*

Proof. 1. Excerise; see Homework 9.

2. If $\mathbf{u} = \beta\mathbf{v}$, then

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \beta\mathbf{v}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v} = \beta\mathbf{v} = \mathbf{u}.$$

Conversely, suppose $\mathbf{u} = \operatorname{proj}_{\mathbf{v}}\mathbf{u}$. Then by definition

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v},$$

so set $\beta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}$ and we have $\mathbf{u} = \beta\mathbf{v}$.

□

Example 7.19. Let $V = \mathcal{C}([-1,1], \mathbb{R})$ be the space of continuous functions on the closed interval $[-1,1]$, with the inner product given as above. Consider the vectors $1, x$. We compute:

$$\begin{aligned}
\|1\| &= \sqrt{\int_{-1}^{1} 1\, dx} = \sqrt{x|_{-1}^{1}} = \sqrt{2} \\
\|x\| &= \sqrt{\int_{-1}^{1} x^2\, dx} = \sqrt{x^3/3|_{-1}^{1}} = \sqrt{2/3} \\
\langle 1, x\rangle &= \int_{-1}^{1} x\, dx = x^2/2|_{-1}^{1} = 0
\end{aligned}$$

so 1 and x are orthogonal. Thus the projection of x onto 1 will give the zero vector: the two vectors have no "direction" in common.

Let's consider now the vector $1 + x$. We have

$$\begin{aligned}
\langle 1 + x, 1\rangle &= \int_{-1}^{1} 1 + x\, dx = x + x^2/2|_{-1}^{1} = 2 \\
\langle 1 + x, x\rangle &= \int_{-1}^{1} x + x^2\, dx = x^2/2 + x^3/3|_{-1}^{1} = 2/3.
\end{aligned}$$

Now we compute

$$\begin{aligned}
\operatorname{proj}_1 1 + x &= \frac{\langle 1 + x, 1\rangle}{\langle 1, 1\rangle}1 = \frac{2}{2}1 = 1 \\
\operatorname{proj}_x 1 + x &= \frac{\langle 1 + x, x\rangle}{\langle x, x\rangle}x = \frac{2/3}{2/3}x = x.
\end{aligned}$$

Thus we can use the inner product to decomopose $1 + x$ into its 1 component and its x component (and the remainder, if there were any).

If two vectors are orthogonal, then they are independent; they don't have any reasonable sub-components pointing in the same direction. This means their lengths are in some sense independent.

Proposition 7.20 (Pythagorean Law)**.** *If* $\mathbf{u}, \mathbf{v}$ *are orthogonal, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Proof. Excercise: See homework 9. □

Example 7.21. Returning to our previous example, we can compute that

$$\|1+x\| = \sqrt{\int_{-1}^{1} 1 + 2x + x^2\, dx} = \sqrt{x + x^2 + x^3/3|_{-1}^{1}} = \sqrt{8/3}.$$

We can confirm that indeed,

$$\|1+x\|^2 = 8/3 = 2 + 2/3 = \|1\| + \|x\|.$$

Using projections we can prove that the Cauchy-Schwarz Inequality, which we saw in theorem 7.8, holds for any inner product.

Theorem 7.22 (Cauchy-Schwarz Inequality)**.** *If* $\mathbf{u}, \mathbf{v}$ *are in an inner product space* V*, then*

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|. \tag{3}$$

Equality holds if and only if $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent.*

Proof. If $\mathbf{v} = \mathbf{0}$, both sides are zero. So assume $\mathbf{v} \neq 0$.

Let $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{u}$. By the Pythagorean law 7.20, we know that

$$\|\mathbf{u}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{u}-\mathbf{p}\|^2.$$

But we know that

$$\|\mathbf{p}\|^2 = \left\| \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}\right\|^2 = \left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\right)^2 \|\mathbf{v}\|^2 = \frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\langle \mathbf{v}, \mathbf{v}\rangle}.$$

Thus we have

$$\begin{aligned}
\frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\langle \mathbf{v}, \mathbf{v}\rangle} &= \|\mathbf{u}\|^2 - \|\mathbf{u}-\mathbf{p}\|^2 \\
\langle \mathbf{u}, \mathbf{v}\rangle^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{p}\|^2\|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \\
|\langle \mathbf{u}, \mathbf{v}\rangle| &\leq \|\mathbf{u}\|\|\mathbf{v}\|.
\end{aligned}$$

Further, we can easily see that we get equality if and only if $\mathbf{u}-\mathbf{p} = \mathbf{0}$, if and only if $\mathbf{u} = \mathbf{p}$, if and only if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. □

Notice that this allows us to define an "angle" between two vectors. Cauchy-Schwarz tells us that

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1,$$

so we can coherently define:

Definition 7.23. If $\mathbf{u}, \mathbf{v}$ are non-zero vectors in an inner product space, we define the angle between them to be

$$\theta = \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

Finally we'd like to return to the idea of distance, by thinking about what properties a "distance" function *should* have. We get

Definition 7.24. A vector space V together with an operation $\| \cdot \| : V \to \mathbb{R}$ is said to be a *normed linear space* if:

1. $\|\mathbf{v}\| \geq 0$ for any $\mathbf{v} \in V$, with equality if and only if $\mathbf{v} = \mathbf{0}$.
2. $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$ for any $\alpha \in \mathbb{R}, \mathbf{v} \in V$.
3. (Triangle inequality) $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ for all $\mathbf{v}, \mathbf{w} \in V$.

Remark 7.25. These three conditions are equivalent to:

1. Nothing has a negative length, and only the zero vector has zero length.
2. Stretching a vector by a scalar multiplies its length by that scalar.
3. The sum of the lengths of two sides of a triangle is greater than the length of the third side. In other words, you can't get somewhere faster by adding a detour in the middle.

A normed linear space is in some sense the right setting in which to do calculus.

Proposition 7.26. *Let V be an inner product space. Then V is a normed linear space with norm given by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.*

Proof. The first two conditions are easy to prove, so we'll just check the triangle inequality.

We compute

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle \\
&\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| && \text{Cauchy-Schwarz Inequality} \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 .
\end{aligned}$$

Taking the square root of both sides gives

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

as desired. □

Remark 7.27. There are norms that do not come from inner products. A good example is the norm on $\mathbb{R}^n$ given by $\|(a_1, a_2, \ldots, a_n\|_1 = |a_1| + |a_2| + \cdots + |a_n|$. We won't worry too much about those in this course, though.

7.3 Orthonormal Bases

Throughout the course, we've been suggesting that we would often like to change from one coordinate system into another which is easier to work with. In this section we'll discuss one particular type of nice basis: one in which all the basis elements are orthogonal.

Definition 7.28. A set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is said to be *orthogonal* if $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$. We say it is *orthonormal* if every vector has magnitude 1.

Proposition 7.29. *Any orthogonal set of non-zero vectors is linearly independent.*

Proof. Suppose

$$a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n = \mathbf{0}.$$

Then dotting the equation with itself, we get

$$\begin{aligned} \langle a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n, a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \rangle &= 0 \\ \sum_{i,j=1}^n a_i a_j \langle \mathbf{u}_i, \mathbf{u}_j \rangle &= 0 \end{aligned}$$

But since the $\mathbf{u}_i$ are orthogonal, $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ when $i \neq j$, so this just gives us

$$\begin{aligned} \sum_{i=1}^n a_i^2 \langle \mathbf{u}_i, \mathbf{u}_i \rangle &= 0 \\ a_1^2\|\mathbf{u}_1\|^2 + \cdots + a_n^2\|\mathbf{u}_n\|^2 &= 0. \end{aligned}$$

And thus, since $\|\mathbf{u}_i\| > 0$, we must have $a_i = 0$ for each i. □

Thus every orthogonal set is a basis for its span.

Definition 7.30. Let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for V. We say that E is an *orthogonal basis* if $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ whenever $i \neq j$.

We say that E is an *orthonormal basis* if, furthermore, $\|\mathbf{e}_i\| = 1$. Thus E is an orthonormal basis if and only if

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Example 7.31.

- The standard basis for $\mathbb{R}^3$ is orthonormal.
- The basis $\{(1,1,0), (1,-1,0), (0,0,1)\}$ for $\mathbb{R}^3$ is orthogonal but not orthonormal.

  But $\{(\sqrt{2}/2, \sqrt{2}/2, 0), (\sqrt{2}/2, -\sqrt{2}/2, 0), (0,0,1)\}$ is orthornormal.
- Let $V = \mathcal{P}_2(x)$ with inner product given by $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)\,dt$. The basis $E = \{1, x, 3x^2 - 1\}$ is an orthogonal basis for V, but not orthonormal.

  The basis $F = \left\{\frac{1}{\sqrt{2}}, \frac{x\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)\right\}$ is orthonormal.
- Let $V = \mathcal{P}_2(x)$ with inner product given by $\langle f, g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$. Then $E = \{1, x, x^2 - 2/3\}$ is an orthogonal basis for V.

  An orthonormal basis is $F = \left\{\frac{\sqrt{3}}{3}, \frac{x\sqrt{2}}{2}, \frac{\sqrt{3}}{\sqrt{2}}\left(x^2 - \frac{2}{3}\right)\right\}$.

Orthonormal bases are particularly nice, for a few reasons.

Proposition 7.32. *Suppose $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for V. Then if $\mathbf{u} = \sum_{i=1}^n a_i \mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^n b_i \mathbf{e}_i$, then $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^n a_i b_i$.*

Consequently $\|\mathbf{u}\| = |a_1|^2 + \cdots + |a_n|^2$.

Remark 7.33. We use this all the time when we're computing the norm of vectors in $\mathbb{R}^n$. This also gives us our "normal" dot product.

Mor importantly, orthonormal bases make projection, coordinates, and changes of basis very easy.

Proposition 7.34. *Let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for V, with $\langle e_i, e_j \rangle = 0$ when $i \neq j$. Then if $\mathbf{v} \in V$, we have*

$$\mathbf{v} = \sum_{i=1}^n \left(\text{proj}_{\mathbf{e}_i} \mathbf{v}\right) \mathbf{e}_i = \left(\text{proj}_{\mathbf{e}_1} \mathbf{v}\right) \mathbf{e}_1 + \cdots + \left(\text{proj}_{\mathbf{e}_n} \mathbf{v}\right) \mathbf{e}_n.$$

Proof. Write $\mathbf{v} = a_1 \mathbf{e}_1 + \cdots + a_n \mathbf{e}_n$ and compute each projection. □

Corollary 7.35. *Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis for V. Then*

$$\mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{e}_i \rangle \mathbf{e}_i = \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \mathbf{e}_n.$$

Example 7.36. $E = \{(1,1,0), (1,-1,0), (0,0,1)\}$ is orthogonal. Find the E coordinates of $(6,2,1)$.

We compute:

$$\begin{aligned}
\operatorname{proj}_{\mathbf{e}_1}(6,2,1) &= \frac{(6,2,1)\cdot(1,1,0)}{(1,1,0)\cdot(1,1,0)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{8}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{e}_1}(6,2,1) &= \frac{(6,2,1)\cdot(1,-1,0)}{(1,-1,0)\cdot(1,-1,0)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{4}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\
\operatorname{proj}_{\mathbf{e}_3}(6,2,1) &= \frac{(6,2,1)\cdot(0,0,1)}{(0,0,1)\cdot(0,0,1)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
[(6,2,1)]_E &= (4,2,1)
\end{aligned}$$

Example 7.37. Let $V = \mathcal{P}_2(x)$, with inner product given by $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$. Then $E = \{1, x, 3x^2 - 1)$ is orthogonal. Write $3x^2 - 6x + 4$ in E-coordinates.

We compute

$$\begin{aligned}
\operatorname{proj}_{\mathbf{e}_1} 3x^2 - 6x + 4 &= \frac{\langle 3x^2 - 6x + 4, 1 \rangle}{\langle 1, 1 \rangle}(1) = \frac{1}{2}\int_{-1}^{1} 3x^2 - 6x + 4\,dx(1) \\
&= \frac{1}{2}\left(x^3 - 3x^2 + 4x \mid |_{-1}^{1}\right)(1) = 5(1) \\
\operatorname{proj}_{\mathbf{e}_2} 3x^2 - 6x + 4 &= \frac{\langle 3x^2 - 6x + 4, x \rangle}{\langle x, x \rangle}(x) = \frac{3}{2}\int_{-1}^{1} 3x^3 - 6x^2 + 4x\,dx(x) \\
&= \frac{3}{2}\left(\frac{x^4}{4} - 2x^3 + 2x^2 \mid |_{-1}^{1}\right)(x) = -6(x) \\
\operatorname{proj}_{\mathbf{e}_3} 3x^2 - 6x + 4 &= \frac{\langle 3x^2 - 6x + 4, 3x^2 - 1 \rangle}{\langle 3x^2 - 1, 3x^2 - 1 \rangle}(3x^2 - 1) \\
&= \frac{5}{8}\int_{-1}^{1}(3x^2 - 6x + 4)(3x^2 - 1)\,dx(3x^2 - 1) \\
&= \frac{5}{8}\left(9x^5/5 - 9x^4/2 + 3x^3 + 3x^2 - 4x \mid |_{-1}^{1}\right)(3x^2 - 1) = 1(3x^2 - 1) \\
\left[3x^2 - 6x + 4\right]_E &= (5, -6, 1).
\end{aligned}$$

Example 7.38. Let $V = \mathbb{R}^3$ with the usual dot product. Then the standard basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ is an orthonormal basis. Use the dot product to find the coordinates of $(2,3,4)$.

We don't need to use the full projection operator; we just need to compute the inner products, since our basis is orthonormal and not just orthogonal.

$$\begin{aligned}
\operatorname{proj}_{\mathbf{e}_1}(2,3,4) &= (2,3,4)\cdot(1,0,0)\begin{bmatrix}1\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix}\\
\operatorname{proj}_{\mathbf{e}_2}(2,3,4) &= (2,3,4)\cdot(0,1,0)\begin{bmatrix}0\\1\\0\end{bmatrix} = 3\begin{bmatrix}0\\1\\0\end{bmatrix}\\
\operatorname{proj}_{\mathbf{e}_3}(2,3,4) &= (2,3,4)\cdot(0,0,1)\begin{bmatrix}0\\0\\1\end{bmatrix} = 4\begin{bmatrix}0\\0\\1\end{bmatrix}
\end{aligned}$$

$$[(2,3,4)]_E = (2,3,4).$$

This isn't a surprise because it was already in coordinates with respect to the standard basis. But this also illustrates a more general principle: if your vector is already written in orthonormal coordinates, your inner product just becomes a dot product.

We'd like a way to generate an orthonormal basis if we don't already have one. This turns out to be straightforward; start with any basis, and one-by-one "fix" elements so that they're orthogonal to all the others.

Proposition 7.39 (Gram-Schmidt Process)**.** *Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis for V. Then there is an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$, where we set:*

$$\begin{aligned}
\mathbf{f}_1 &= \mathbf{e}_1 & \mathbf{u}_1 &= \frac{\mathbf{f}_1}{\|\mathbf{f}_1\|}\\
\mathbf{f}_2 &= \mathbf{e}_2 - \operatorname{proj}_{\mathbf{u}_1}\mathbf{e}_2 & \mathbf{u}_2 &= \frac{\mathbf{f}_2}{\|\mathbf{f}_2\|}\\
\mathbf{f}_3 &= \mathbf{e}_3 - \left(\operatorname{proj}_{\mathbf{u}_1}\mathbf{e}_3 + \operatorname{proj}_{\mathbf{u}_2}\mathbf{e}_3\right) & \mathbf{u}_3 &= \frac{\mathbf{f}_3}{\|\mathbf{f}_3\|}\\
&\vdots & &\vdots\\
\mathbf{f}_n &= \mathbf{e}_n - (\operatorname{proj}_{\mathbf{u}_1}\mathbf{e}_n + \cdots + \operatorname{proj}_{\mathbf{u}_{n-1}}\mathbf{e}_n) & \mathbf{u}_n &= \frac{\mathbf{f}_n}{\|\mathbf{f}_n\|}.
\end{aligned}$$

Proof. It's clear that each $\mathbf{u}_i$ has norm 1, so we just need to check that they are pairwise orthogonal, which is the same as checking that the $\mathbf{f}_i$ are all orthogonal

But we have constructed the $\mathbf{f}_i$ to be orthogonal by subtracting off the pieces they have in common. For instance, we see that

$$\begin{aligned}\langle \mathbf{f}_1, \mathbf{f}_2 \rangle &= \left\langle \mathbf{e}_1, \mathbf{e}_2 - \frac{\langle \mathbf{e}_2, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \mathbf{e}_1 \right\rangle = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle - \frac{\langle \mathbf{e}_2, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle \\ &= \langle \mathbf{e}_1, \mathbf{e}_2 \rangle - \langle \mathbf{e}_2, \mathbf{e}_1 \rangle = 0.\end{aligned}$$

In general, we see that

$$\langle \mathbf{f}_j, \operatorname{proj}_{\mathbf{f}_j} \mathbf{f}_i \rangle = \left\langle \mathbf{f}_i, \frac{\langle \mathbf{f}_j, \mathbf{f}_j \rangle}{\langle \mathbf{f}_j, \mathbf{f}_j \rangle} \mathbf{f}_j \right\rangle = \langle \mathbf{f}_i, \mathbf{f}_j \rangle$$

and all the other projections will be zero since the $\mathbf{f}_i$ are orthogonal, so each $\mathbf{f}_j$ is orthogonal to all the previous $\mathbf{f}_i$. □

Example 7.40. Let $V = \mathbb{R}^3$ with the usual dot product, and let $E = \{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$. Use Gram-Schmidt to orthonormalize this basis.

We take $\mathbf{f}_1 = (1, 1, -1)$, and then we compute $\|\mathbf{f}_1\| = \sqrt{3}$ so $\mathbf{u}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right)$.

Then we set

$$\begin{aligned}\mathbf{f}_2 &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \operatorname{Proj}_{(1,1,-1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{(1, -1, 1) \cdot (1, 1, -1)}{(1, -1, 1) \cdot (1, -1, 1)} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{-1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -2/3 \\ 2/3 \end{bmatrix} \\ \mathbf{u}_2 &= \frac{\mathbf{f}_2}{\|\mathbf{f}_2\|} = \frac{(4/3, -2/3, 2/3)}{\sqrt{24/9}} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 4/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/3 \\ -\sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}.\end{aligned}$$

Finally we have

$$\begin{aligned}
\mathbf{f}_3 &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \operatorname{proj}_{(1,1,-1)} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \operatorname{proj}_{(4,-2,2)} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-1,1,1)\cdot(1,1,-1)}{(1,1,-1)\cdot(1,1,-1)} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{(-1,1,1)\cdot(4,-2,2)}{(4,-2,2)\cdot(4,-2,2)} \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{-1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{-4}{24} \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
\mathbf{u}_3 &= \frac{\mathbf{f}_3}{\|\mathbf{f}_3\|} = \frac{(0,1,1)}{\sqrt{2}} = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}.
\end{aligned}$$

Thus an orthonormal basis for $\mathbb{R}^3$ is

$$\left\{ \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ -\sqrt{3}/3 \end{bmatrix}, \begin{bmatrix} \sqrt{6}/3 \\ -\sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\}.$$

Notice that while this is an orthonormal basis for $\mathbb{R}^3$, it is not the usual one. We will get different orthonormal bases out of the end, depending on which vector we start with.

Example 7.41. Let $V = \mathcal{P}_2(x)$ with the inner product given by $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. (Note this is a *different* inner product from the one we've been using!) Let's form an orthonormal basis from the set $\{1, x, x^2\}$.

We set $\mathbf{f}_1 = 1$. We compute that

$$\|1\|^2 = \langle 1, 1\rangle = \int_0^1 1\,dx = 1$$

so this is already a unit vector; we set $\mathbf{u}_1 = 1$.

We take

$$\mathbf{f}_2 = x - \frac{\langle 1, x\rangle}{\langle 1, 1\rangle}(1) = x - \frac{1}{2}(1) = x - 1/2.$$

We compute

$$\|\mathbf{f}_2\| = \sqrt{\int_0^1 (x-1/2)^2\,dx} = \sqrt{12} = 2\sqrt{3}$$

so we set

$$\mathbf{u}_2 = \frac{\mathbf{f}_2}{\|\mathbf{f}_2\|} = 2\sqrt{3}(x - 1/2) = \sqrt{3}(2x - 1).$$

Finally, we have

$$\begin{aligned}
\mathbf{f}_3 &= x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle \sqrt{3}(2x-1), x^2\rangle}{\langle \sqrt{3}(2x-1), \sqrt{3}(2x-1)\rangle}\sqrt{3}(2x-1) \\
&= x^2 - \int_0^1 x^2\,dx(1) - \sqrt{3}\int_0^1 2x^3 - x^2\,dx(\sqrt{3}(2x-1)) \\
&= x^2 - \frac{1}{3} - \frac{1}{2}(2x-1) = x^2 - x + \frac{1}{6}.
\end{aligned}$$

Then we compute

$$\begin{aligned}
\|\mathbf{f}_3\| &= \sqrt{\int_0^1 (x^2 - x + 1/6)^2\,dx} = \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}} \\
\mathbf{u}_3 &= \frac{\mathbf{f}_3}{\|\mathbf{f}_3\|} = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}.
\end{aligned}$$

Thus an orthonormal basis for $\mathcal{P}_2(x)$ with this inner product is

$$\{1, \sqrt{3}(2x-1), \sqrt{5}(6x^2 - 6x + 1)\}.$$

7.4 Orthogonal Subspaces

We have used orthogonality to give a vector space a particularly nice basis. We can also break the vector space into two (or more) independent subspaces.

Definition 7.42. If V is an inner product space and U, W are subspaces, we say that U and W are *orthogonal* and write $U \perp W$ if $\langle \mathbf{u}, \mathbf{w}\rangle = 0$ for every $\mathbf{u} \in U, \mathbf{w} \in W$.

If $U \subset V$, we define the *orthogonal complement of U* to be the set of all vectors perpendicular to everything in U:

$$U^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{u}\rangle = 0 \forall \mathbf{u} \in U\}.$$

Example 7.43.

- In $\mathbb{R}^2$, the orthogonal complement of a line is a line. The orthogonal complement to a set with two points in it is also a line.
- In $\mathbb{R}^3$, the orthogonal complement of a line is a plane, and the orthogonal complement of a plane is a line.

Proposition 7.44. *If U is a subset of V, then $U^\perp$ is a subspace of V.*

Proof. 1. $\mathbf{0}$ is orthogonal to everything, and thus is in $U^\perp$.

2. Suppose $\mathbf{v} \in U^\perp$, and $r \in \mathbb{R}$. Then for any $\mathbf{u} \in U$ we have $\langle r\mathbf{v}, \mathbf{u}\rangle = r\langle \mathbf{v}, \mathbf{u}\rangle = r \cdot 0 = 0$, so $r\mathbf{v} \in U^\perp$ by definition.

3. Suppose $\mathbf{v}, \mathbf{w} \in U^\perp$, and let $\mathbf{u} \in U$. Then

$$\langle \mathbf{v} + \mathbf{w}, \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{w}, \mathbf{u}\rangle = 0 + 0 = 0.$$

Thus $\mathbf{v} + \mathbf{w}$ is orthogonal to $\mathbf{u}$ for every $\mathbf{u} \in U$, and so $\mathbf{v} + \mathbf{w} \in U^\perp$.

Thus by the subspace theorem, $\mathbf{U}^\perp$ is a subspace of V. □

Remark 7.45. We will usually consider cases where U is also a subspace of V, but this isn't necessary; nothing above assumes anything about the structure of U.

A basic thing we want to do is, given a subspace, find a basis for the subspace and for its orthogonal complement. As with everything else, we can solve this problem by row-reducing matrices.

Proposition 7.46. *Let A be a matrix. Then* $\ker(A) = (\mathrm{row}(A))^\perp$.

Remark 7.47. In three dimensions, we can use this exact formula to find the normal vector to a plane.

Proof. If $\mathbf{r}_i$ are the rows of the matrix A, then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix}$$

and thus $\mathbf{x} \in \ker(A)$ precisely if $\mathbf{x}$ is orthogonal to every row of A. But if $\mathbf{x}$ is orthogonal to every row vector of A, it is orthogonal to every linear combination of them, and thus is orthogonal to their span, which is the rowspace. □

Example 7.48. Suppose we want to find the orthogonal complement to $U = \mathrm{span}\{(1, 4, 2), (1, 1, 1)\}$. Then we write down the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 3 & 1 \end{bmatrix}$$

so $U^\perp = \ker(A) = \{(-2\alpha, -\alpha, 3\alpha)\} = \mathrm{span}\{(2, 1, -3)\}$. We can check that this is in fact orthogonal to the original two vectors.

There are a couple more useful facts we'd like to know about orthogonal complements, which show that they relate spaces in useful ways.

Proposition 7.49. *If U is a subspace of V and $\mathbf{v} \in V$, then there exist unique $\mathbf{v}_U \in U, \mathbf{v}_{U^\perp} \in U^\perp$ such that $\mathbf{v} = \mathbf{v}_U + \mathbf{v}_{U^\perp}$.*

We say that this is an orthogonal decomposition *of* $\mathbf{v}$.

Proof. Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthogonal basis for U and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_m\}$ an orthogonal basis for $U^\perp$.

We claim that $E \cup F = \{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_m\}$ is an orthogonal basis for V. It must be orthogonal since E and F are orthogonal sets, and thus it is linearly independent. So we need to show that it spans V.

Suppose $\mathbf{v} \in V$, and consider the element

$$\mathbf{v}' = \mathbf{v} - \sum_{i=1}^n \operatorname{proj}_{\mathbf{e}_i} \mathbf{v}.$$

This is an element of V, and by construction it is orthogonal to every $\mathbf{e}_i$ and thus all of U, so $\mathbf{v}' \in U^\perp$. Thus $\mathbf{v}' \in \operatorname{span}(F)$ and so $\mathbf{v} \in \operatorname{span}(E \cup F)$. Thus $E \cup F$ spans V.

Then every element of V can be expressed uniquely as a linear combination of elements of E and F. This gives us a unique representation as a sum of an element of U and an element of $U^\perp$.

□

Corollary 7.50. $\dim U + \dim U^\perp = \dim V$.

Example 7.51. Give the orthogonal decomposition of $(3, -1, 2)$ with respect to the subspace given by $x - y + 2z = 0$ and its complement.

We need to find an orthonormal basis for either $x - y + 2z = 0$ or its orthogonal complement. But we can see that the normal vector to this plane is in the orthogonal complement, so $\{(1, -1, 2)\}$ is a basis for $U^\perp$.

We project $(3, -1, 2)$ onto $\operatorname{span}\{(1, -1, 2)\}$. We have

$$\operatorname{proj}_{(1,-1,2)} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \frac{(3,-1,2) \cdot (1,-1,2)}{(1,-1,2) \cdot (1,-1,2)} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{8}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -4/3 \\ 8/3 \end{bmatrix}$$

So this is the projection into $U^\perp$. The projection into U then is just what's left over: it's

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \operatorname{proj}_{(1,-1,2)} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 - 4/3 \\ -1 + 4/3 \\ 2 - 8/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$$

(We check that this vector is in fact in the plane U). Then we have an orthogonal decomposition: $(3, -1, 2) = (5/3, 1/3, -2/3) + (4/3, -4/3, 8/3)$.

Example 7.52. Let $V = \mathbb{R}^4$ and let $U = \operatorname{span}\{(1, 2, 3, 4), (2, 1, -1, -2)\}$. Find the orthogonal decomposition of $(1, 1, 1, 1)$ into its components in U and $U^\perp$.

We write a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -5 & -8 \\ 0 & 3 & 7 & 10 \end{bmatrix}$$

so $\ker(A) = (5\alpha + 8\beta, -7\alpha - 10\beta, 3\alpha, 3\beta) = \operatorname{span}\{(5, -7, 3, 0), (8, -10, 0, 3)\}$.

We need to find an orthogonal basis for either U or $U^\perp$.We compute

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} - \operatorname{proj}_{(1,2,3,4)} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} - \frac{(2, 1, -1, -2) \cdot (1, 2, 3, 4)}{(1, 2, 3, 4) \cdot (1, 2, 3, 4)} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} - \frac{-7}{30} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 67 \\ 44 \\ -9 \\ -32 \end{bmatrix}$$

We compute

$$\operatorname{proj}_{\mathbf{f}_1}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{(1,2,3,4)\cdot(1,1,1,1)}{(1,2,3,4)\cdot(1,2,3,4)}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \frac{10}{30}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\1\\4/3\end{bmatrix}$$

$$\operatorname{proj}_{\mathbf{f}_2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{(1,1,1,1)\cdot(67,44,-9,-32)}{(67,44,-9,-32)\cdot(67,44,-9,-32)}\begin{bmatrix}67\\44\\-9\\-32\end{bmatrix} = \frac{70}{7530}\begin{bmatrix}67\\44\\-9\\-32\end{bmatrix} = \frac{7}{753}\begin{bmatrix}67\\44\\-9\\-32\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}_U = \begin{bmatrix}1/3\\2/3\\1\\4/3\end{bmatrix} + \frac{7}{753}\begin{bmatrix}67\\44\\-9\\-32\end{bmatrix} = \frac{10}{251}\begin{bmatrix}24\\27\\23\\26\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}_{U^\perp} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\end{bmatrix}_U = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{10}{251}\begin{bmatrix}24\\27\\23\\26\end{bmatrix} = \frac{1}{251}\begin{bmatrix}11\\-19\\21\\-9\end{bmatrix}.$$

Proposition 7.53. *If U is a subspace of V, then $(U^\perp)^\perp = U$.*

Proof. If $\mathbf{u} \in U$, then $\mathbf{u}$ is orthogonal to every $\mathbf{w} \in U^\perp$ by definition. So $U \subset (U^\perp)^\perp$.

Conversely, suppose $\mathbf{w} \in (U^\perp)^\perp$. We can write $\mathbf{w} = \mathbf{w}_U + \mathbf{w}_{U^\perp}$. Then $\mathbf{w} \in (U^\perp)^\perp$ so we know $\langle \mathbf{w}, \mathbf{w}_{U^\perp}\rangle = \mathbf{0}$.

But $\langle \mathbf{w}, \mathbf{w}_{U^\perp}\rangle = \langle \mathbf{w}_{U^\perp}, \mathbf{w}_{U^\perp}\rangle = 0$ if and only if $\mathbf{w}_{U^\perp} = 0$. Thus $\mathbf{w}_{U^\perp} = \mathbf{0}$, and $\mathbf{w} = \mathbf{w}_U \in U$. □