Math 214 Spring 2020
Linear Algebra HW 10
Due Tuesday, April 28

For all these problems, justify your answers; do not just write "yes" or "no" or give a single number.

1. Compute e^A and e^B, where

$$A = \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}.$$

2. The employees at a certain company can choose to sign up for a health benefit or not each year. Past data indicates that 80% of those who are enrolled one year will stay enrolled the next year, and 40% of those who are not enrolled will choose to enroll the next year.

   (a) Write down a matrix for this Markov process.

   (b) If 50% of the company is enrolled this year, how many employees will be enrolled next year? The year after that?

   (c) In the long run, what percentage of employees do you expect to be enrolled in the health benefit?

3. ($\star$) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ be the matrix of a Markov process.

   (a) Is there a steady state vector? That is, can you find a vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$, and the entries of $\mathbf{v}$ sum to one? What is it?

   (b) Suppose you start with the distribution vector $\mathbf{w} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$. What distribution do you get after five steps? Ten? Fifteen?

   (c) Is this Markov process guaranteed to converge to a steady state? Why or why not?

4. Let $\mathbf{u} = (2, 1, 3)$ and $\mathbf{v} = (6, 3, 9)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

(c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.

5. Let $\mathbf{u} = (2, -5, 4)$ and $\mathbf{v} = (1, 2, -1)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.

6. Let $\mathbf{u} = (4, 1)$ and $\mathbf{v} = (3, 2)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.

7. Let $\mathbf{u} = (3, 5)$ and $\mathbf{v} = (1, 1)$.

   (a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   (c) Verify that $\mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$.

8. Let $V = \mathcal{P}_n(x)$ and fix real numbers $x_0, x_1, \ldots, x_n$ be distinct real numbers. For $f, g \in V$, define

   $$\langle f, g \rangle = \sum_{i=0}^{n} f(x_i)g(x_i).$$

   Prove this is an inner product on V.

   (Hint: See partial proof from class)

9. Let $w_1, \ldots, w_n$ be positive real numbers. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define

   $$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i w_i.$$

   Prove that this is an inner product on $\mathbb{R}^n$. (The w_i are called the *weights* of the inner product).

10. Let $V = \mathcal{C}([1, 3], \mathbb{R})$, with the usual inner product $\langle f, g \rangle = \int_1^3 f(t)g(t)\,dt$. Find $\|1\|$ and $\|x\|$. Find the projection of $1 + x$ onto 1 and x.

11. Prove the Pythagorean law: if $\mathbf{u}, \mathbf{v}$ are orthogonal, then $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$.

12. Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space V, with $\mathbf{v} \neq 0$. Let $\mathbf{p} = \mathrm{proj}_{\mathbf{v}}\,\mathbf{u}$. (Hint: compare the end of section 6.1).

    Prove that $\langle \mathbf{u} - \mathbf{p}, \mathbf{p} \rangle = 0$.