

Math 214 Spring 2020  
 Linear Algebra HW 1 Solutions  
 Due Thursday, January 30

For all these problems, justify your answers.

1. Find all solutions to the system

$$\begin{aligned}x_1 + 2x_2 - 3x_3 &= 9 \\2x_1 - x_2 + x_3 &= 0 \\4x_1 - x_2 + x_3 &= 4.\end{aligned}$$

**Solution:**

$$\begin{aligned}\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & -9 & 13 & -32 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 2/5 & 2/5 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 1 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array} \right]\end{aligned}$$

Thus the solution is  $x_1 = 2, x_2 = 5, x_3 = 1$ .

2. Find all solutions to the system

$$\begin{aligned}2x_1 + 3x_2 - x_3 + 4x_4 &= 1 \\3x_1 - x_2 + x_4 &= 1 \\3x_1 - 4x_2 + x_3 - x_4 &= 2.\end{aligned}$$

**Solution:**

$$\begin{aligned}\left[ \begin{array}{cccc|c} 2 & 3 & -1 & 4 & 1 \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array} \right] &\rightarrow \left[ \begin{array}{cccc|c} 1 & 3/2 & -1/2 & 2 & 1/2 \\ 3 & -1 & 0 & 1 & 1 \\ 3 & -4 & 1 & -1 & 2 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 3/2 & -1/2 & 2 & 1/2 \\ 0 & -11/2 & 3/2 & -5 & -1/2 \\ 0 & -17/2 & 5/2 & -7 & 1/2 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{cccc|c} 1 & 3/2 & -1/2 & 2 & 1/2 \\ 0 & 1 & -3/11 & 10/11 & 1/11 \\ 0 & -17/2 & 5/2 & -7 & 1/2 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & -1/11 & 7/11 & 4/11 \\ 0 & 1 & -3/11 & 10/11 & 1/11 \\ 0 & 0 & 2/11 & 8/11 & 14/11 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & -1/11 & 7/11 & 4/11 \\ 0 & 1 & -3/11 & 10/11 & 1/11 \\ 0 & 0 & 1 & 4 & 7 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 & 7 \end{array} \right].\end{aligned}$$

Thus we set the free variable  $x_4 = \alpha$  and our set of solutions is

$$\{(1 - \alpha, 2 - 2\alpha, 7 - 4\alpha, \alpha) = \{(1, 2, 7, 0) + \alpha(-1, -2, -4, 1)\}.$$

3. Find all solutions to the system

$$\begin{aligned} 5x_1 + 3x_2 - x_3 &= 3 \\ 3x_1 - 3x_2 + x_3 &= 2 \\ -x_1 + 2x_2 + 4x_3 &= 1 \\ x_1 - x_2 + 3x_3 &= 0. \end{aligned}$$

**Solution:**

$$\begin{aligned} \left[ \begin{array}{ccc|c} 5 & 3 & -1 & 3 \\ 3 & -3 & 1 & 2 \\ -1 & 2 & 4 & 1 \\ 1 & -1 & 3 & 0 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & 3 & -1 & 3 \\ 3 & -3 & 1 & 2 \\ -1 & 2 & 4 & 1 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 8 & -16 & 3 \\ 0 & 0 & -8 & 2 \\ 0 & 1 & 7 & 1 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 7 & 1 \\ 0 & 8 & -16 & 3 \\ 0 & 0 & -8 & 2 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 10 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & -72 & -5 \\ 0 & 0 & -8 & 2 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 10 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & -72 & -5 \end{array} \right] &\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 11/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 13 \end{array} \right]. \end{aligned}$$

Since the last equation is  $0 = 13$  this system is inconsistent and has no solutions.

4. Reduce the following matrix to reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \left[ \begin{array}{ccccc} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{array} \right] &\rightarrow \left[ \begin{array}{ccccc} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{array} \right] &\rightarrow \left[ \begin{array}{ccccc} 1 & 2 & -4 & -4 & 5 \\ 0 & 1 & -10 & -9 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccccc} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -10 & -9 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & 25 \end{array} \right] &\rightarrow \left[ \begin{array}{ccccc} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -10 & -9 & 5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & 25 \end{array} \right] &\rightarrow \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 54 \end{array} \right] \\ &\rightarrow \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] &\rightarrow \left[ \begin{array}{ccccc} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \end{aligned}$$

5. Reduce the following matrix to reduced row echelon form:

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix}.$$

**Solution:**

$$\begin{aligned} & \begin{bmatrix} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix} \\ \rightarrow & \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & -2 & -1 & 0 & -4 & 5 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \\ \rightarrow & \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \rightarrow & \begin{bmatrix} 1 & 0 & 1/4 & 1 & 21/4 & -1 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

6. Compute

$$\begin{aligned} & \begin{bmatrix} 6 & 3 & 10 \\ 4 & 8 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1.5 \\ 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 28 & 38 \\ 41 & 40 \end{bmatrix} \\ & \begin{bmatrix} 1 & 3 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 3 & -1 \\ 5 & -2 & -1 & 1 \\ -1 & 2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 12 & -8 & 0 & -4 \\ 2 & 4 & -5 & 7 \end{bmatrix} \end{aligned}$$

7. Compute:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -9 & -8 \\ -6 & -4 & 5 \\ 5 & 0 & -4 \end{bmatrix} \\ & \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 3 & -1 \\ 6 & -1 & -4 \\ -11 & 6 & 4 \end{bmatrix} \end{aligned}$$

8. (★) Let  $A$  be a  $n \times n$  matrix. If  $A$  is invertible, prove that  $A^{-1}$  is unique. In other words, if  $AB = I_n = CA$ , prove that  $B = C$ .

(Hint: what is the matrix  $CAB$  equal to?)

**Solution:** Since  $AB = I_n$  we know that  $CAB = C(AB) = CI_n = C$ . But since  $CA = I_n$  we know that  $CAB = (CA)B = I_n B = B$ . Thus  $B = C$ .