Math 214 Spring 2020
Linear Algebra HW 2 Solutions
Due Friday, Thursday, February 6

For all these problems, justify your answers.

1. Suppose A is a matrix such that $A^{-1} = \begin{bmatrix} 3 & 1 & 5 \\ 2 & -1 & 5 \\ 1 & 4 & -3 \end{bmatrix}$. Find all solutions to $A\mathbf{x} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

   (Do not try to actually compute the matrix A.) **Solution:**

$$\mathbf{x} = A^{-1} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 5 \\ 2 & -1 & 5 \\ 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 19 \end{bmatrix}.$$

2. Find the inverse of $\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ or prove it is not invertible.

Solution:

$$\left[\begin{array}{cccc|cccc} 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 5 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 5 & -6 & 2 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 5 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & -6 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 & 11 & 2 & -4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 3 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & -6 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 & -5 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 11/3 & 2/3 & -4/3 & 1/3 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -13/3 & -1/3 & 5/3 & -2/3 \\ 0 & 1 & 0 & 0 & 4/3 & 1/3 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 7/3 & 1/3 & -2/3 & 2/3 \\ 0 & 0 & 0 & 1 & 11/3 & 2/3 & -4/3 & 1/3 \end{array}\right].$$

Thus we have

$$A^{-1} = \begin{bmatrix} -13/3 & -1/3 & 5/3 & -2/3 \\ 4/3 & 1/3 & -2/3 & 2/3 \\ 7/3 & 1/3 & -2/3 & 2/3 \\ 11/3 & 2/3 & -4/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -13 & -1 & 5 & -2 \\ 4 & 1 & -2 & 2 \\ 7 & 1 & -2 & 2 \\ 11 & 2 & -4 & 1 \end{bmatrix}.$$

3. Find the inverse of $\begin{bmatrix} 3 & 2 & 1 & 5 \\ 2 & 4 & 3 & 8 \\ -1 & 2 & 5 & 4 \\ 4 & 8 & 9 & 17 \end{bmatrix}$ or prove it is not invertible.

Solution:

$$\left[\begin{array}{cccc|cccc} 3 & 2 & 1 & 5 & 1 & 0 & 0 & 0 \\ 2 & 4 & 3 & 8 & 0 & 1 & 0 & 0 \\ -1 & 2 & 5 & 4 & 0 & 0 & 1 & 0 \\ 4 & 8 & 9 & 17 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 3 & 2 & 1 & 5 & 1 & 0 & 0 & 0 \\ 2 & 4 & 3 & 8 & 0 & 1 & 0 & 0 \\ 4 & 8 & 9 & 17 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 16 & 17 & 1 & 0 & 3 & 0 \\ 0 & 8 & 13 & 16 & 0 & 1 & 2 & 0 \\ 0 & 16 & 29 & 33 & 0 & 0 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 16 & 17 & 1 & 0 & 3 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 0 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & -2 & -5 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 16 & 17 & 1 & 0 & 3 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right].$$

We get a row of all zeroes on the left-hand block, so the matrix is not invertible.

4. Find the nullspace $\begin{bmatrix} 3 & -2 & 2 & -5 \\ 1 & 0 & -2 & -2 \\ -4 & 2 & -4 & 3 \end{bmatrix}$. (Express your answer as a set).

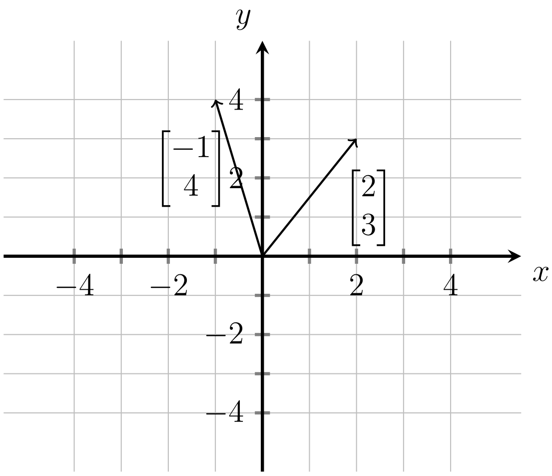
Solution:

$$\begin{bmatrix} 3 & -2 & 2 & -5 \\ 1 & 0 & -2 & -2 \\ -4 & 2 & -4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & -2 & 8 & 1 \\ 0 & 2 & -12 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 2 & -8 & -1 \\ 0 & 0 & -4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus the nullspace is the set $\{(0, 7/2\alpha, -\alpha, \alpha)\}$. Alternatively, you could write $N(A) = \{(0, 7\alpha, -2\alpha, 2\alpha)\}$.

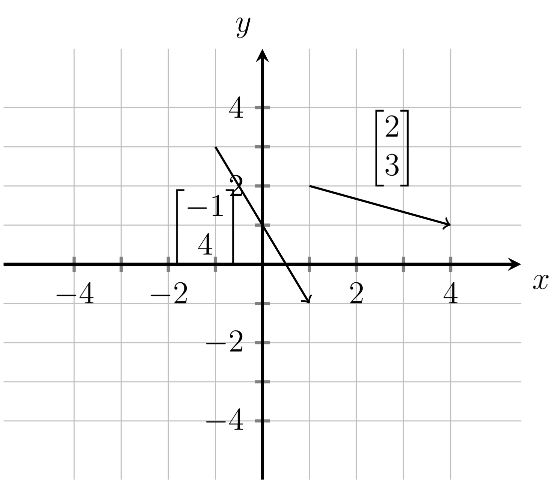
5. (a) Draw a graph of the Cartesian plane with $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ in standard position.

Solution:



(b) Draw a graph of the Cartesian plane with the vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ with its tail at the point $(1, 2)$, and the vector $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ with its tail at $(-1, 3)$.

Solution:



6. Use the picture below to:

(a) Write the vector $\overrightarrow{AB}$ in standard vector notation.

Solution: $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$.

(b) Write the vector $\mathbf{v}$ in standard vector notation.

Solution: $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) Find the vector $\mathbf{u} + \mathbf{w}$ and write it in standard vector notation. **Solution:** $\mathbf{u} + \mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

7. (a) If $A = (2, 1)$ and $B = (-2, 2)$, write the vector $\overrightarrow{AB}$ in standard vector notation.

   Solution: $\overrightarrow{AB} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

   (b) If $C = (1, -1, 0)$ and $D = (0, 1, 2)$, write the vector $\overrightarrow{CD}$ in standard vector notation. **Solution:** $\overrightarrow{CD} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

8. Compute the following:

   (a)

$$\begin{bmatrix} 1 \\ -3/2 \\ 4 \end{bmatrix} + \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1/2 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 5 \\ 3 \\ 7 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ -3 \\ 1 \\ \pi \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 4 \\ 7+\pi \\ 4 \end{bmatrix}$$

9. Compute the following:

$$e \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2e \\ e \\ -2e \\ -3e \end{bmatrix} \qquad -3 \cdot \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \\ -3 \end{bmatrix}$$

10. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$, let $\mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$, and let $\mathbf{w} = \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}$.

    (a) Compute $2\mathbf{v} + 3\mathbf{u}$

    Solution:

$$2\mathbf{v} + 3\mathbf{u} = \begin{bmatrix} 8 \\ -4 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 23 \end{bmatrix}.$$

    (b) Compute $5\mathbf{u} + 2\mathbf{w}$.

    Solution:

$$5\mathbf{u} + 2\mathbf{w} = \begin{bmatrix} -5 \\ 0 \\ 15 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ -6 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \\ 9 \end{bmatrix}.$$