

Math 214 Spring 2020
Linear Algebra HW 3
Due Thursday, February 13

1. (★) Let $\mathcal{P}(x) = \{a_0 + a_1x + \cdots + a_nx^n : n \in \mathbb{N}, a_i \in \mathbb{R}\}$ be the set of polynomials with real coefficients. Define addition by

$$(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and define scalar multiplication by

$$r(a_0 + a_1x + \cdots + a_nx^n) = ra_0 + ra_1x + \cdots + ra_nx^n.$$

Prove that $\mathcal{P}(x)$ is a vector space.

2. Prove that if $r\mathbf{u} = \mathbf{0}$, then either $r = 0$ or $\mathbf{u} = \mathbf{0}$.
3. (★) Show that the zero vector is unique. That is, if \mathbf{v} is a vector with the property that $\mathbf{v} + \mathbf{u} = \mathbf{u}$ for every vector $\mathbf{u} \in V$, then $\mathbf{v} = \mathbf{0}$.
4. (a) Show that the set $\{(x, x, y, y) | x, y \in \mathbb{R}\}$ is a subspace of \mathbb{R}^4 .
(b) Show that the set $\{(x, y, 0) | x, y \in \mathbb{R}\}$ is a subspace of \mathbb{R}^3 . What does this subspace look like geometrically?
(c) Show that the set $\{(x, 2x, 3x) | x \in \mathbb{R}\}$ is a subspace of \mathbb{R}^3 . What does this subspace look like geometrically?
5. (a) Show that the set $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(0) = 0\}$ is a vector space. (Hint: Show it is a subspace of something we know is a vector space).
(b) Show that the set $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(0) = 1\}$ is not a vector space.
6. (a) Show that if n is a positive integer, then the set $\mathcal{P}_n(x)$ of polynomials of degree at most n is a vector space.
(b) Show that the set $\mathcal{C}(\mathbb{R}, \mathbb{R})$ the set of continuous functions of one real variable is a vector space.
7. Which of the following are vector spaces? You don't need to justify your answers.
- (a) $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(0) = 0 \text{ and } f(1) = 0\}$
(b) $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(0) = 0 \text{ or } f(1) = 0\}$
(c) $\{f : \mathbb{R} \rightarrow \mathbb{R} | f \text{ is constant}\}$

(d) $\mathcal{C}([a, b], \mathbb{R})$ the space of continuous functions from $[a, b]$ to \mathbb{R} .

8. Which of the following are vector spaces? You don't need to justify your answers.

(a) $\{(a, b) \in \mathbb{R}^2 \mid a + b = 0\}$

(b) $\{(a, b) \in \mathbb{R}^2 \mid a + b = 3\}$

(c) $\{a_0 + a_1x + a_2x^2 \in \mathcal{P}_2(x) \mid a_1 = 1\}$.

(d) $\{a_0 + a_2x^2 + a_3x^3 + a_5x^5 \in \mathcal{P}_5(x)\}$.