Math 214 Spring 2020
Linear Algebra HW 3 Solutions
Due Thursday, February 13

1. ($\star$) Let $\mathcal{P}(x) = \{a_0 + a_1x + \cdots + a_nx^n : n \in \mathbb{N}, a_i \in \mathbb{R}\}$ be the set of polynomials with real coefficients. Define addition by

$$(a_0+a_1x+\cdots+a_nx^n)+(b_0+b_1x+\cdots+b_nx^n) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

and define scalar multiplication by

$$r(a_0 + a_1x + \cdots + a_nx^n) = ra_0 + ra_1x + \cdots + ra_nx^n.$$

Prove that $\mathcal{P}(x)$ is a vector space.

Solution:

Let $r, s \in \mathbb{R}$ be scalars, and $f(x) = a_0 + \cdots + a_nx^n$, $g(x) = b_0 + \cdots + b_nx^n$, $h(x) = c_0 + \cdots + c_nx^n$ be elements of $\mathcal{P}(x)$. Then

(a)

$$f(x)+g(x) = (a_0 + \cdots + a_nx^n)+(b_0 + \cdots + b_nx^n) = (a_0+b_0)+\cdots+(a_n+b_n)x^n \in \mathcal{P}(x).$$

(b)

$$\begin{aligned} f(x) + g(x) &= (a_0 + \cdots + a_nx^n) + (b_0 + \cdots + b_nx^n) = (a_0 + b_0) + \cdots + (a_n + b_n)x^n \\ &= (b_0 + \cdots + b_nx^n) + (a_0 + \cdots + a_nx^n) = g(x) + f(x). \end{aligned}$$

(c)

$$\begin{aligned} (f(x) + g(x)) + h(x) \quad & ((a_0 + \cdots + a_nx^n) + (b_0 + \cdots + b_nx^n)) + (c_0 + \cdots + c_nx^n) \\ &= ((a_0 + b_0) + \cdots + (a_n + b_n)x^n) + (c_0 + \cdots + c_nx^n) \\ &= ((a_0 + b_0 + c_0) + \cdots + (a_n + b_n + c_0)x^n) \\ &= (a_0 + \cdots + a_nx^n) + ((b_0 + c_0) + \cdots + (b_n + c_n)x^n) \\ &= (a_0 + \cdots + a_nx^n) + ((b_0 + \cdots + b_nx^n) + (c_0 + \cdots + c_nx^n)) \\ &= f(x) + (g(x) + h(x)). \end{aligned}$$

(d) We set $\mathbf{0} = 0$ the zero polynomial. Then we see that

$$0 + f(x) = 0 + (a_0 + \cdots + a_nx^n) = (a_0 + 0) + \cdots + a_nx^n = a_0 + \cdots + a_nx^n = f(x)$$

so we have an additive identity.

(e) Set $-f(x) = (-a_0) + \cdots + (-a_n)x^n$. Then

$$f(x) + (-f(x)) = (a_0 + (-a_0)) + \cdots + (a_n + (-a_n))x^n = 0 + \cdots + 0x^n = 0.$$

(f)

$$rf(x) = r(a_0 + \cdots + a_n x^n) = ra_0 + \cdots + (ra_n)x^n \in \mathcal{P}(x)$$

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(g)

$$\begin{aligned} r(f(x) + g(x)) &= r((a_0 + b_0) + \cdots + (a_n + b_n)x^n) \\ &= (r(a_0 + b_0)) + \cdots + (r(a_n + b_n))x^n \\ &= (ra_0 + rb_0) + \cdots + (ra_n + rb_n)x^n \\ &= (ra_0 + \cdots + ra_n x^n) + (rb_0 + \cdots + rb_n x^n) \\ &= rf(x) + rg(x). \end{aligned}$$

(h)

$$\begin{aligned} (r+s)f(x) &= (r+s)a_0 + \cdots + ((r+s)a_n)x^n \\ &= ra_0 + sa_0 + \cdots + (ra_n + sa_n)x^n \\ &= (ra_0 + \cdots + ra_n x^n) + (sa_0 + \cdots + sa_n x^n) \\ &= r(a_0 + \cdots + a_n x^n) + s(a_0 + \cdots + a_n x^n) = rf(x) + sf(x). \end{aligned}$$

(i)

$$\begin{aligned} (rs)f(x) &= rsa_0 + \cdots + (rsa_n)x^n = r(sa_0) + \cdots + (r(sa_n))x^n \\ &= r(sa_0 + \cdots + (sa_n)x^n) = r(sf(x)). \end{aligned}$$

(j)

$$1f(x) = 1(a_0 + \cdots + a_n x^n) = 1a_0 + \cdots + (1a_n)x^n = a_0 + \cdots + a_n x^n = f(x).$$

2. Prove that if $r\mathbf{u} = \mathbf{0}$, then either $r = 0$ or $\mathbf{u} = \mathbf{0}$.

Solution: Suppose $r\mathbf{u} = \mathbf{0}$ and $r \neq 0$. Then $1/r$ is a real number, and we can compute

$$\begin{aligned} \frac{1}{r}(r\mathbf{u}) &= \frac{1}{r}\mathbf{0} && \\ \left(\frac{1}{r}\right) \cdot r)\mathbf{u}\frac{1}{r}\mathbf{0} & && \text{multiplicative associativity} \\ 1 \cdot \mathbf{u} &= \frac{1}{r}\mathbf{0} && \text{arithmetic} \\ \mathbf{u} &= \frac{1}{r}\mathbf{0} && \text{multiplicative identity} \\ \mathbf{u} &= \mathbf{0} && \text{proposition from class.} \end{aligned}$$

Thus if $r \neq 0$ then $\mathbf{u} = \mathbf{0}$.

3. ($\star$) Show that the zero vector is unique. That is, if $\mathbf{v}$ is a vector with the property that $\mathbf{v} + \mathbf{u} = \mathbf{u}$ for every vector $\mathbf{u} \in V$, then $\mathbf{v} = \mathbf{0}$.

Solution: Here are two different proofs:

First, we can simply suppose $\mathbf{v} + \mathbf{u} = \mathbf{u}$ for some $\mathbf{v}, \mathbf{u} \in V$. Then

$$\begin{aligned} \mathbf{v} + \mathbf{u} &= \mathbf{u} + \mathbf{0} && \text{additive identity} \\ \mathbf{v} &= \mathbf{0} && \text{cancellation.} \end{aligned}$$

Possibly slightly better is to observe that if $\mathbf{0}$ is not unique, then there is some $\hat{\mathbf{0}}$ with the same property. Then we have

$$\begin{aligned} \mathbf{0} + \hat{\mathbf{0}} &= \hat{\mathbf{0}} && \text{additive identity} \\ \mathbf{0} + \hat{\mathbf{0}} &= \mathbf{0} && \text{given property} \\ \mathbf{0} &= \hat{\mathbf{0}} && \text{transitive property.} \end{aligned}$$

4. (a) Show that the set $\{(x, x, y, y) | x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^4$.
   (b) Show that the set $\{(x, y, 0) | x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. What does this subspace look like geometrically?
   (c) Show that the set $\{(x, 2x, 3x) | x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. What does this subspace look like geometrically?

   Solution:

   (a) We need to check the three subspace properties.
      i. The zero vector $(0, 0, 0, 0)$ is in this set, corresponding to $x = y = 0$.
      ii. If $(x_1, x_1, y_1, y_1), (x_2, x_2, y_2, y_2)$ are in the set, then

      $$(x_1, x_1, y_1, y_1) + (x_2, x_2, y_2, y_2) = (x_1 + x_2, x_1 + x_2, y_1 + y_2, y_1 + y_2)$$

      is also in the set (corresponding to $x = x_1 + x_2, y = y_1 + y_2$).
      iii. If $r \in \mathbb{R}$ and (x_1, x_1, y_1, y_1) is in the set, then

      $$r(x_1, x_1, y_1, y_1) = (rx_1, rx_1, ry_1, ry_1)$$

      is in the set (corresponding to $x = rx_1, y = ry_1$).

      Thus it is a subspace by the subspace proposition.

   (b) We check the three subspace properties.
      i. The zero vector $(0, 0, 0)$ is in this set, corresponding to $x = y = 0$.
      ii. If $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ are in the set, then

      $$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0)$$

      is in the set, corresponding to $x = x_1 + x_2, y = y_1 + y_2$.
      iii. If $r \in \mathbb{R}$ and $(x_1, y_1, 0)$ is a vector in the set, then

      $$r(x_1, y_1, 0) = (rx_1, ry_1, 0)$$

      is in the set, corresponding to $x = rx_1, y = ry_1$.

This is the xy plane through the origin.

(c) We need to check the three subspace properties.

i. The zero vector $(0, 0, 0)$ is in the set.

ii. If $(x, 2x, 3x)$ and $(y, 2y, 3y)$ are in the set, then

$$(x, 2x, 3x) + (y, 2y, 3y) = (x + y, 2(x + y), 3(x + y))$$

is also in the set.

iii. If $r \in \mathbb{R}$ and $(x, 2x, 3x)$ is a vector in the set, then

$$r(x, 2x, 3x) = (rx, 2(rx), 3(rx))$$

is also a vector in the set.

Thus it is a subspace by the subspace proposition. The set is a line through the origin (angling up).

5. (a) Show that the set $\{f : \mathbb{R} \to \mathbb{R} | f(0) = 0\}$ is a vector space. (Hint: Show it is a subspace of something we know is a vector space).

(b) Show that the set $\{f : \mathbb{R} \to \mathbb{R} | f(0) = 1\}$ is not a vector space.

Solution:

(a) This is a set of functions, so it is a subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. So we just need to check the three subspace properties.

i. If $\mathbf{0}$ is the zero function given by $\mathbf{0}(x) = 0$, then $\mathbf{0}(0) = 0$ and thus $\mathbf{0}$ is in the set.

ii. Suppose f, g are functions with $f(x) = 0 = g(x)$. Then $(f + g)(x) = f(x) + g(x) = 0 + 0 = 0$, so $f + g$ is in the set.

iii. Suppose f is a function with $f(x) = 0$ and $r \in \mathbb{R}$. Then $(rf)(0) = rf(0) = r0 = 0$ so rf is in the set.

Thus the set is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ by the subspace proposition.

(b) We just need to show that one of the three subspace properties fails. In this case, we see that $\mathbf{0}(0) = 0 \neq 1$, so the zero vector is not contained in this set. Thus it is not a vector space.

6. (a) Show that if n is a positive integer, then the set $\mathcal{P}_n(x)$ of polynomials of degree at most n is a vector space.

(b) Show that the set $\mathcal{C}(\mathbb{R}, \mathbb{R})$ the set of continuous functions of one real variable is a vector space.

Solution:

(a) We know that $\mathcal{P}_n(x) \subseteq \mathcal{P}(x)$ the space of all polynomials, so we just have to check the three subspace properties.

i. The zero vector is the zero polynomial $0 = 0 + 0x + \cdots + 0x^n \in \mathcal{P}_n(x)$.

ii. If $a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_nx^n \in \mathcal{P}_n(x)$, then

$$a_0+a_1x+\cdots+a_nx^n+b_0+b_1x+\cdots+b_nx^n \in \mathcal{P}_n(x) = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n \in \mathcal{P}_n($$

iii. If $a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}_n(x)$ then

$$r\left(a_0 + a_1 x + \cdots + a_n x^n\right) = (ra_0) + (ra_1)x + \cdots + (ra_n)x^n \in \mathcal{P}_n(x).$$

Thus $\mathcal{P}_n(x)$ is a subspace of $\mathcal{P}(x)$ by the subspace proposition.

(b) We know that $\mathcal{C}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, R)$, so we just need to check the three subspace properties.

i. The zero function $f(x) = 0$ is continuous, so $\mathbf{0} \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.

ii. If $f, g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ then f, g are continuous functions, so $f + g$ is a continuous function (from calculus 1) and thus $f + g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$.

iii. If $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $r \in \mathbb{R}$ then f is a continuous function, and thus rf is a continuous function and thus in $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

Thus $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

7. Which of the following are vector spaces? You don't need to justify your answers.

(a) $\{f : \mathbb{R} \to \mathbb{R} | f(0) = 0 \text{ and } f(1) = 0\}$

(b) $\{f : \mathbb{R} \to \mathbb{R} | f(0) = 0 \text{ or } f(1) = 0\}$

(c) $\{f : \mathbb{R} \to \mathbb{R} | f \text{ is constant}\}$

(d) $\mathcal{C}([a, b], \mathbb{R})$ the space of continuous functions from $[a, b]$ to $\mathbb{R}$.

Solution:

(a) yes

(b) no

(c) yes

(d) yes

8. Which of the following are vector spaces? You don't need to justify your answers.

(a) $\{(a, b) \in \mathbb{R}^2 | a + b = 0\}$

(b) $\{(a, b) \in \mathbb{R}^2 | a + b = 3\}$

(c) $\{a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2(x) | a_1 = 1\}$.

(d) $\{a_0 + a_2 x^2 + a_3 x^3 + a_5 x^5 \in \mathcal{P}_5(x)\}$.

Solution:

(a) yes

(b) no

(c) no

(d) yes