Math 214 Spring 2020
Linear Algebra HW 4 Solutions
Due Thursday, February 27

1. (a) Write $x + x^2 + x^3$ as a linear combination of $x, x + 2x^2, x^2 - 4x^3$.

   Solution: We want to solve the equation $ax+b(x+2x^2)+c(x^2-4x^3) = x+x^2+x^3$. This gives us $(a + b)x + (2b + c)x^2 - 4cx^3 = x + x^2 + x^3$, so we have the system

$$\begin{aligned} a + b &= 1 \\ 2b + c &= 1 \\ -4c &= 1 \end{aligned}$$

   which gives the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 5/4 \\ 0 & 0 & 1 & -1/4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 5/8 \\ 0 & 0 & 1 & -1/4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/8 \\ 0 & 1 & 0 & 5/8 \\ 0 & 0 & 1 & -1/4 \end{array}\right].$$

   Thus we see that

$$x + x^2 + x^3 = \frac{3}{8}x + \frac{5}{8}(x + 2x^2) - \frac{1}{4}(x^2 - 4x^3).$$

   (b) Write $4x + 6x^3 - x^5$ as a linear combination of $x + x^3, x^3 + x^5$, and $x + x^5$.

   Solution: We can do this by trial and error, or systematically. The most systematic approach is to write the equations

$$4x + 6x^2 - x^5 = a(x + x^3) + b(x^3 + x^5) + c(x + x^5) = (a + c)x + (a + b)x^3 + (b + c)x^5$$

   and thus we get

$$4 = a + c \qquad 6 = a + b \qquad -1 = b + c.$$

   We have $a = 4-c$, and thus $b = 6-a = 6-4+c = c+2$, and thus $-1 = b+c = 2c+2$ and thus $c = -3/2$. Then $a = 11/2$ and $b = 1/2$. So we have

$$4x + 6x^2 - x^5 = 11/2(x + x^3) + 1/2(x^3 + x^5) - 3/2(x + x^5).$$

2. Let $V = \mathbb{R}^3$.

(a) Is $S = \{(1,2,3),(2,3,4),(3,4,5)\}$ a spanning set for $\mathbb{R}^3$?

Solution: We try to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \\ 3\alpha_1 + 4\alpha_2 + 5\alpha_3 \end{bmatrix}$$

and get the system of equations

$$a = \alpha_1 + 2\alpha_2 + 3\alpha_3 \qquad b = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 \qquad c = 3\alpha_1 + 4\alpha_2 + 5\alpha_3.$$

We get

$$\begin{aligned}
\alpha_1 &= a - 2\alpha_2 - 3\alpha_3 \\
3\alpha_2 &= b - 2\alpha_1 - 4\alpha_3 = b - 2(a - 2\alpha_2 - 3\alpha_3) - 4\alpha_3 \\
&= b - 2a + 4\alpha_2 + 2\alpha_3 \\
\alpha_2 &= 2a - b - 2\alpha_3 \\
5\alpha_3 &= c - 3\alpha_1 - 4\alpha_2 = c - 3(a - 2\alpha_2 - 3\alpha_3) - 4\alpha_2 \\
&= c - 3a + 2\alpha_2 + 9\alpha_3 = c - 3a + 2(2a - b - 2\alpha_3) + 9\alpha_3 \\
&= c + a - 2b + 5\alpha_3 \\
0 &= c + a - 2b
\end{aligned}$$

Thus any vector in the span must have $2b = a + c$, and so the set does not span V.

(b) Is $T = \{(1,2,3),(2,3,4),(0,1,1)\}$ a spanning set for $\mathbb{R}^3$?

Solution: We try to solve

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 + 3\alpha_2 + \alpha_3 \\ 3\alpha_1 + 4\alpha_2 + \alpha_3 \end{bmatrix}$$

and get the system of equations

$$a = \alpha_1 + 2\alpha_2 \qquad b = 2\alpha_1 + 3\alpha_2 + \alpha_3 \qquad c = 3\alpha_1 + 4\alpha_2 + \alpha_3.$$

We get

$$\begin{aligned}
\alpha_1 &= a - 2\alpha_2 \\
3\alpha_2 &= b - 2\alpha_1 - \alpha_3 = b - 2(a - 2\alpha_2) - 4\alpha_3 \\
&= b - 2a + 4\alpha_2 - 4\alpha_3 \\
\alpha_2 &= 2a - b + 4\alpha_3 \\
\alpha_3 &= c - 3\alpha_1 - 4\alpha_2 = c - 3(a - 2\alpha_2) - 4\alpha_2 \\
&= c - 3a + 2\alpha_2 = c - 3a + 2(2a - b + 4\alpha_3) \\
&= c + a - 2b + 4\alpha_3 \\
\alpha_3 &= 2/3b - 1/3a - 1/3c.
\end{aligned}$$

Since these equations have a solution, T is a spanning set for $\mathbb{R}^3$.

3. (a) Is $S = \{(1,1,0,0),(1,-1,0,0),(0,0,1,-1),(0,0,-1,1)\}$ a spanning set for $\mathbb{R}^4$?

Solution: We solve

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 \\ \alpha_3 - \alpha_4 \\ -\alpha_3 + \alpha_4 \end{bmatrix}$$

which gives us the system of equations

$$\alpha_1 + \alpha_2 = a \qquad \alpha_1 - \alpha_2 = b \qquad \alpha_3 - \alpha_4 = c \qquad \alpha_4 - \alpha_3 = d$$

and we see that we must have $c = -d$, so the set does not span.
(We see that we would have $\alpha_1 = (a+b)/2$ and $\alpha_2 = (a-b)/2$).

(b) Is $T = \{1, 1+x, 1+x^2\}$ a spanning set for $\mathcal{P}_2(x)$?

Solution: We want to solve

$$a + bx + cx^2 = \alpha_0(1) + \alpha_1(1+x) + \alpha_2(1+x^2) = (\alpha_0 + \alpha_1 + \alpha_2) + \alpha_1 x + \alpha_2 x^2$$

which gives us the system

$$a = \alpha_0 + \alpha_1 + \alpha_2 \qquad b = \alpha_1 \qquad c = \alpha_2$$

which has the solution

$$\alpha_2 = c \qquad \alpha_1 = b \qquad \alpha_0 = a - b - c.$$

Thus this is a spanning set.

4. Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ is a spanning set for V. Prove that $T = \{\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2, \ldots, \mathbf{v}_n - \mathbf{v}_{n-1}\}$ is a spanning set for V.

Solution: Suppose we have an element $\mathbf{u} \in V$. Since S is a spanning set for V, this means that $\mathbf{u}$ is a linear combination of elements of S, so there exist $b_1, \ldots, b_n$ such that

$$\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

so we want to solve the equation

$$\begin{aligned} a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n &= \alpha_1\mathbf{v}_1 + \alpha_2(\mathbf{v}_2 - \mathbf{v}_1) + \cdots + \alpha_n(\mathbf{v}_n - \mathbf{v}_{n-1}) \\ &= (\alpha_1 - \alpha_2)\mathbf{v}_1 + \cdots + (\alpha_{n-1} - \alpha_n)\mathbf{v}_{n-1} + \alpha_n\mathbf{v}_n \end{aligned}$$

which gives us the system

$$a_1 = \alpha_1 - \alpha_2 \qquad \ldots \qquad a_{n-1} = \alpha_{n-1} - \alpha_n \qquad a_n = \alpha_n$$

and we can solve this to get

$$\begin{aligned} \alpha_n &= a_n & \qquad \alpha_{n-1} &= a_{n-1} + \alpha_n = a_{n-1} + a_n \\ &\ldots & \alpha_1 &= a_1 + a_2 + a_3 + \cdots + a_n. \end{aligned}$$

Thus $\mathbf{u} \in \operatorname{span}(T)$. Since $\mathbf{u}$ was an arbitrary vector in V, this means that T spans V.

5. (a) Is $S = \{(1,1,1),(1,1,0),(1,0,0)\}$ a linearly independent set?

Solution: Suppose

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a+b+c\\b+c\\c\end{bmatrix}.$$

Then we have the system

$$a+b+c=0 \qquad b+c=0 \qquad c=0$$

from which we see that $c=0$, and thus $b=0-c=0$ and $a=0-b-c=0-0-0=0$. Thus S is linearly independent.

(b) Is $T = \{(1,2,3),(4,5,6),(7,8,9)\}$ a linearly independent set?

Solution: Suppose

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}4\\5\\6\end{bmatrix} + c\begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}a+4b+7c\\2a+5b+8c\\3a+6b+9c\end{bmatrix}.$$

Then we have the system

$$a+4b+7c=0 \qquad 2a+5b+8c=0 \qquad 3a+6b+9c=0.$$

Then we have

$$\begin{aligned}a &= -4b-7c\\ 5b &= -2a-8c = -2(-4b-7c)-8c = 8b+6c\\ b &= -2c\\ a &= 8c-7c = c\\ c &= -a/3-2b/3 = -c/3+4c/3.\end{aligned}$$

In particular we see that if $c=1, a=1, b=-2$ then we have a solution to this system.

Alternatively, we can simply notice that

$$\begin{bmatrix}7\\8\\9\end{bmatrix} = (-1)\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}4\\5\\6\end{bmatrix}$$

and thus the set is not linearly independent, since we can write one element as a linear combination of the others.

(c) Is $U = \{(3,7,5),(2,4,2),(1,3,1)\}$ a linearly independent set?

Solution: Suppose

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = a\begin{bmatrix}3\\7\\5\end{bmatrix} + b\begin{bmatrix}2\\4\\2\end{bmatrix} + c\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}3a+2b+c\\7a+4b+3c\\5a+2b+c\end{bmatrix}.$$

Then we have the system

$$3a + 2b + c = 0 \qquad 7a + 4b + 3c = 0 \qquad 5a + 2b + c = 0.$$

We can subtract the first equation from the third to see $2a = 0$ and thus $a = 0$. Then we have $c = -2b$ and $4b = -3c = 6b$ so $b = 0$ and then $c = 0$. So U is linearly independent.

6. (a) Is $S = \{1 + x, 1 + x^2, x + x^2\}$ a linearly independent set?

Solution: Suppose

$$0 = a(1 + x) + b(1 + x^2) + c(x + x^2) = (a + b) + (a + c)x + (b + c)x^2.$$

Then we have the system

$$0 = a + b \qquad 0 = a + c \qquad 0 = b + c.$$

This tells us that $a = -b$ from the first equation, and thus $b = c$ from the second, and thus $2c = 0$ from the third. Thus $c = 0$, and then $b = 0$ and $a = 0$. So S is linearly independent.

(b) Is $T = \{1 + x, 1 + x^2, x - x^2\}$ a linearly independent set?

Solution: Suppose

$$0 = a(1 + x) + b(1 + x^2) + c(x - x^2) = (a + b) + (a + c)x + (b - c)x^2.$$

This gives us the system

$$0 = a + b \qquad 0 = a + c \qquad 0 = b - c.$$

This gives us $b = c$, and then the other two equations become the same; so we see that if $c = 1$ then $b = 1, a = -1$ is a solution to the system.

Alternatively, we can notice that

$$(1 + x) - (1 + x^2) = (1 - 1) + x - x^2 = x - x^2$$

so T is not linearly independent because one of the vectors can be written as a linear combination of the others.

(c) Is $U = \{\sin^2, \cos^2, 1\}$ a linearly independent set?

Solution: We don't have an easy way to turn this into a system of linear equations. But we can notice (or recall from class) that $\sin^2 + \cos^2 = 1$. Thus U is not linearly independent, since one vector can be written as a linear combination of the others.

7. ($\star$) Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent in V, and $T = \{\mathbf{v}_1 + \mathbf{w}, \dots, \mathbf{v}_n + \mathbf{w}\}$ is linearly dependent in V. Prove that $\mathbf{w} \in \operatorname{span}(S)$.

Solution: Since T is linearly dependent, we have constants $a_1, \dots, a_n$ not all zero such that

$$0 = a_1(\mathbf{v}_1 + \mathbf{w}) + \cdots + a_n(\mathbf{v}_n + \mathbf{w}).$$

We can rearrange this equation to give

$$-(a_1 + \cdots + a_n)\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

We would like to divide by $-(a_1 + \cdots + a_n)$, but we may only do this if we know $a_1 + \cdots + a_n \neq 0$, so we need to prove that somehow. So suppose for contradiction that $a_1 + \cdots + a_n = 0$. Then our previous equation becomes

$$\mathbf{0} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

Since S is linearly independent, we know that $a_1 = \cdots = a_n = 0$; but by hypothesis we know that some a_i is nonzero, which is a contradiction. Thus we must have $a_1 + \cdots + a_n \neq 0$.

Then we have the equation

$$\mathbf{w} = \frac{-a_1}{a_1 + \cdots + a_n}\mathbf{v}_1 + \cdots + \frac{-a_n}{a_1 + \cdots + a_n}\mathbf{v}_n.$$

Then we have written $\mathbf{w}$ as a linear combination of vectors in S, so $\mathbf{w} \in \operatorname{span}(S)$.

8. Prove that a set $S = \{\mathbf{u}, \mathbf{v}\}$ of two vectors is linearly dependent if and only if one is a scalar multiple of the other.

   Solution: Suppose S is linearly dependent. Then there is some solution to $\mathbf{0} = a\mathbf{u} + b\mathbf{v}$ where the coefficients are not both zero; without loss of generality assume $a \neq 0$. Then we can write $\mathbf{u} = -b/a\mathbf{v}$ and thus one vector is a scalar multiple of the other.

   Conversely, suppose $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, that is, suppose there is a scalar a with $\mathbf{u} = a\mathbf{v}$. Then we can write $\mathbf{0} = (-1)\mathbf{u} + a\mathbf{v}$. Since $-1 \neq 0$, we have written $\mathbf{0}$ as a nontrivial linear combination of $\mathbf{u}$ and $\mathbf{v}$, so S is linearly dependent.