Math 214 Spring 2020
Linear Algebra HW 6
Due Thursday, March 19

For all these problems, justify your answers; do not just write "yes" or "no".

1. Let $L : U \to V$ be a linear transformation. Prove that $\ker(L)$ is a subspace of U.

2. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$L((x, y, z)) = \begin{bmatrix} x + y + z \\ 3x - 2y + z \\ 2z \end{bmatrix}.$$

   (a) Prove that L is a linear transformation.

   (b) Write a matrix for L.

   (c) Find a basis for the kernel.

   (d) Find a basis for the image.

3. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define $E_0 : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ to be the function given by $E_0(f) = f(0)$. (Thus if $f(x) = x^3 + 3x + 1$ then $E_0(f) = f(0) = 1$).

   (a) Prove that E_0 is a linear transformation.

   (b) What is the kernel of E_0? What is the image?

4. Let $P_Z : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map onto the xy plane, given by $P(x, y, z) = (x, y, 0)$.

   (a) Prove that P_z is a linear transformation.

   (b) Find a matrix for P_z.

   (c) Find a basis for the kernel and image of P_z.

   (d) Prove that $P_z(P_z(\mathbf{u})) = P_z(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^3$. Linear transformations with this property are called *projections* and we will revisit them later. (They are also sometimes called *idempotent* if you're feeling particularly fancy).

5. Let $L : \mathbb{R} \to \mathbb{R}$ be a linear transformation. Prove that there is some real number $r \in \mathbb{R}$ such that $L(x) = rx$ for all $x \in \mathbb{R}$. (In other words, any linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is given by multiplication by a scalar).

6. ($\star$) Let $L : U \to V$ be a linear transformation, and let $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for U. Prove that $L(U) = \text{span}(L(E))$. That is, prove the image of L is just the span of the image of E under L.

7. Let $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$. Find bases for the rowspace, columnspace, and nullspace of A.

8. Let $B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$. Find bases for the rowspace, columnspace, and nullspace of B.

9. Use Gaussian elimination to find a basis for the span of $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$.

10. For each of the following systems of equations, is there a solution? You don't need to find the solution if it exists, but justify your answer. (Hint: think about the column space).

    (a)
    $$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ?$$

    (b)
    $$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 1 \end{bmatrix} ?$$