Math 214 Test 1
Practice Problem Solutions

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

Solve the following systems of linear equations

1.

$$\begin{aligned} x - 4y + 2z &= 2 \\ -x + 3y + z &= 4 \\ 2x - y + z &= 1 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & -4 & 2 & 2 \\ -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -4 & 2 & 2 \\ 0 & -1 & 3 & 6 \\ 0 & 7 & -3 & -3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -10 & -22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 18 & 39 \end{array}\right]$$
$$\to \left[\begin{array}{rrr|r} 1 & 0 & -10 & -22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 1 & 13/6 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & -2/6 \\ 0 & 1 & 0 & 3/6 \\ 0 & 0 & 1 & 13/6 \end{array}\right]$$

so we have $x = -1/3, y = 1/2, z = 13/6$.

2.

$$\begin{aligned} x + 3y + 7z &= 15 \\ 2x + 9y + 23z &= 45 \\ x - z &= 2 \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 3 & 7 & 15 \\ 2 & 9 & 23 & 45 \\ 1 & 0 & -1 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 7 & 15 \\ 0 & 3 & 9 & 15 \\ 0 & -3 & -8 & -13 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 3 & 9 & 15 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\to \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

So the system has one solution, which is $x = 4, y = -1, z = 2$.

3.

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ x_1 - 5x_2 + x_4 &= 5 \end{aligned}$$

Solution: $\left\{\left(\frac{15}{4}-\frac{5\alpha}{8}-\beta, \frac{-1}{4}-\frac{\alpha}{8}, \alpha, \beta\right)\right\}$

4.

$$\begin{aligned} -x_1+2x_2-x_3&=2\\ -2x_1+2x_2+x_3&=4\\ 3x_1+2x_2+2x_3&=5\\ -3x_1+8x_2+5x_3&=17 \end{aligned}$$

Solution: $\{(0, 3/2, 1)\}$.

5.

$$\begin{aligned} x-2y&=3\\ 2x+y&=1\\ -5x+8y&=4 \end{aligned}$$

Solution: No solution exists.

Do the following matrix multiplication computations.

1.

$$\begin{bmatrix} 1 & 0 & -3\\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 5\\ 3 & 1\\ -1 & 2 \end{bmatrix}=$$

Solution:

$$\begin{bmatrix} 1 & 0 & -3\\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 5\\ 3 & 1\\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 5 & -1\\ 8 & 25 \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & 5\\ 3 & 1\\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3\\ 3 & 2 & 4 \end{bmatrix}=$$

Solution:

$$\begin{bmatrix} 2 & 5\\ 3 & 1\\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3\\ 3 & 2 & 4 \end{bmatrix}=\begin{bmatrix} 17 & 10 & 14\\ 6 & 2 & -5\\ 5 & 4 & 11 \end{bmatrix}$$

3.

$$\begin{bmatrix} 5 & 2 & -1\\ 1 & 0 & 3\\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} -3 & 0 & 1\\ 4 & -1 & 1\\ 0 & 2 & 3 \end{bmatrix}=$$

Solution:

$$\begin{bmatrix} 5 & 2 & -1\\ 1 & 0 & 3\\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} -3 & 0 & 1\\ 4 & -1 & 1\\ 0 & 2 & 3 \end{bmatrix}=\begin{bmatrix} -7 & -4 & 4\\ -3 & 6 & 10\\ -4 & 6 & 18 \end{bmatrix}$$

4.

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 2 & 3 & 8 \end{bmatrix} =$$

Solution: Undefined, since the dimensions are wrong.

For each of the following matrices, find:

(a) The reduced row echelon form.

(b) The nullspace

1. $\begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so the nullspace is the trivial space $\{\mathbf{0}\}$.

2. $\begin{bmatrix} -2 & 4 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is $\{\alpha/6, -\alpha/6, \alpha\}$

3. $\begin{bmatrix} 6 & 2 & 3 & 1 \\ 1 & 5 & 2 & -2 \\ 4 & -4 & 1 & 3 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & -1/5 \end{bmatrix}$, so the nullspace is $\{(-2\alpha/5, 2\alpha/5, \alpha/5, \alpha)\}$.

4. $\begin{bmatrix} -1 & 3 & 4 \\ 2 & 5 & 2 \\ 0 & 1 & 3 \\ 4 & 1 & -2 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so the nullspace is trivial.

Find the inverses of the following matrices, or show they are not invertible.

1. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

   Solution: We form the augmented matrix $\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$ and row-reduce to get $\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

2. $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

Solution: We notice that the two rows are linearly dependent, so the rank is 1 and the nullity is 1, and the matrix has no inverse.

Alternatively, we form the augmented matrix $\left[\begin{array}{cc|cc} 3 & 1 & 0 & 1 \\ 6 & 2 & 1 & 0 \end{array}\right]$ and row-reduce to get $\left[\begin{array}{cc|cc} 1 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$. Thus the matrix has no inverse.

3. $\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \\ 1 & -4 & 3 \end{bmatrix}$

Solution: We form the augmented matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 1 & -4 & 3 & 0 & 0 & 1 \end{array}\right]$ and row-reduce to get $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/27 & 8/27 & 4/27 \\ 0 & 1 & 0 & 5/27 & 1/54 & -13/54 \\ 0 & 0 & 1 & 7/27 & -2/27 & -1/27 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} -1/27 & 8/27 & 4/27 \\ 5/27 & 1/54 & -13/54 \\ 7/27 & -2/27 & -1/27 \end{bmatrix}$.

4. $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 2 \\ 7 & 1 & 5 \end{bmatrix}$

Solution: We may notice that the third row is the sum of the first two, and thus the matrix has rank 2 and nullity 1, so is not invertible.

Alternatively, we form the augmented matrix $\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 & 1 & 0 \\ 7 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$ and row-reduce to get $\left[\begin{array}{ccc|ccc} 1 & 0 & 2/5 & 0 & 1/5 & 0 \\ 0 & 1 & 11/5 & 0 & -7/5 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$. Thus there is no inverse.

Which of the following are vector spaces? Prove or disprove your answer, potentially using the subspace theorem

1. $\{(a, b, c, d) : a - b = c - d\}$
2. $\{(a, b, c, d) : a + b + c = d\}$
3. $\{(a, b, c) : a^2 = bc\}$
4. $\{(a, b, c, d) : 5a - 3b = 2c - 2d\}$
5. $\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_2 = 2\}$
6. $\{f(x) : f(0) = 5\}$
7. $\{f(x) : f(5) = 0\}$

Solution:

1. Yes
2. Yes
3. No: $(0, 1, 0)$ and $(0, 0, 1)$ are in the set, but $(0, 1, 0) + (0, 0, 1) = (0, 1, 1)$ is not.
4. Yes
5. No: $2x^2$ is in the set, but $2 \cdot 2x^2 = 4x^2$ is not.

6. No: the constant function $f(x) = 5$ is in the set, but $(2f)(x) = 10$ so $2f$ is not in the set.
7. Yes.

Write u as a linear combination of vectors in S, or prove you cannot

1. $\mathbf{u} = (5, 2, 1)$, $S = \{(1, 2, 3), (3, 1, 1)\}$

   Solution: Not possible

2. $\mathbf{u} = (2, 3, 2)$, $S = \{(1, 2, 3), (3, 4, 1)\}$

   Solution: $\mathbf{u} = \frac{1}{2}(1, 2, 3) + \frac{1}{2}(3, 4, 1)$.

3. $\mathbf{u} = x^3 - x + 1$, $S = \{1 + x, 3 + x^2, 3x^2 + x^3\}$

   Solution: Not possible

4. $\mathbf{u} = x^3 + 4x^2 + 2x + 5$, $S = \{1 + x, 3 + x^2, 3x^2 + x^3\}$

   Solution: $2(1 + x) + (3 + x^2) + (3x^2 + x^3) = 5 + 2x + 2x^3 + x^3$.

Proofs

1. Suppose U, W are subspaces of some vector space V. Prove that the set $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}$ is a subspace of V.

   Bonus: what is the space $U + U$?

   Solution: We know that $\mathbf{0} \in U$ and $\mathbf{0} \in W$, so $\mathbf{0} + \mathbf{0} = \mathbf{0} \in U + W$.

   Suppose $\mathbf{u}_1 + \mathbf{w}_1, \mathbf{u}_2 + \mathbf{w}_2 \in U + W$. Then $(\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{w}_1 + \mathbf{w}_2) \in U + W$ since $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{w}_1 + \mathbf{w}_2 \in W$ by closure under addition of vector spaces.

   Suppose $\mathbf{u} + \mathbf{w} \in U + W$, and $r \in \mathbb{R}$. Then $r(\mathbf{u} + \mathbf{w}) = r\mathbf{u} + r\mathbf{w} \in U + W$ since $r\mathbf{u} \in U$ and $r\mathbf{w} \in W$ by closure under scalar multiplication of vector spaces.

   Thus by the subspace theorem, $U + W$ is a subspace.

   Bonus: $U + U = U$, essentially because U is closed under addition.

2. Let $A \in M_{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that if $N(A) = \{\mathbf{0}\}$ then the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

   Solution: There are two approaches.

   One is to recall the theorem from class: let's assume the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}_0$. Then the set of solutions looks like $\{\mathbf{x}_0 + \mathbf{u} : \mathbf{u} \in N(A)\}$. But since $N(A) = \{\mathbf{0}\}$ then the set of solutions is $\{\mathbf{x}_0 + \mathbf{0}\} = \{\mathbf{x}_0\}$ and thus there is only one solution.

   If we don't remember that theorem, we can prove it directly. Suppose $A\mathbf{x} = \mathbf{b} = A\mathbf{y}$. Then $A\mathbf{x} - A\mathbf{y} = 0$, so $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, and thus $\mathbf{x} - \mathbf{y} \in N(A)$.

   But $N(A) = \{\mathbf{0}\}$ so this implies that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, and thus $\mathbf{x} = \mathbf{y}$. Thus any two solutions are equal, so there is at most one solution.

Bonus to stretch your brain

1. Find a subset $U \subset \mathbb{R}^2$ that is closed under scalar multiplication but is not a subspace.

   Solution: One possible example is $\{(x, y) : x^2 = y^2\}$. If $x^2 = y^2$ then $(ax)^2 = (ay)^2$ so it's closed under scalar multiplication. But $(1, 1) + (1, -1) = (2, 0)$ is not in this subset, even though $(1, 1)$ and $(1, -1)$ are.

2. Find a subset $U \subset \mathbb{R}^2$ that is closed under addition but is not a subspace.

   Solution: One possible answer is $\mathbb{Z}^2 = \{(a, b) : a, b \in \mathbb{Z}\}$ the set of ordered pairs of integers. If $(a, b), (x, y) \in \mathbb{Z}^2$ then $(a + x, b + y) \in \mathbb{Z}^2$. But we see that while $(1, 0) \in \mathbb{Z}^2$, $.5 \cdot (1, 0) = (.5, 0)$ is not.