Math 214 Test 2
Practice Problems

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This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I expect the test to be five pages covering a subset of this material.

Proofs

1. Let $L : U \to V$ be a linear transformation of vector spaces. Prove that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.
2. Let $L : U \to V$ be a linear transformation.
   (a) If $\dim U > \dim V$, prove that L is not injective. (There is "too much" in U to fit it all in V without repeating).
   (b) If $\dim U < \dim V$, prove that L is not surjective. (There is "not enough" in U to cover all of V).
   (c) Find counterexamples to the converses of these statements. That is, find a function $L : U \to V$ where L is not injective, but $\dim U < \dim V$. And find a function $L : U \to V$ where L is not surjective, but $\dim U > \dim V$.
3. Let $L : \mathbb{R}^4 \to \mathbb{R}^2$ such that $\ker(L) = \{(x, y, z, w) : x = y, z = w\}$. Prove that the image of L is $\mathbb{R}^2$.
4. Let $A \in M_{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that if $N(A) = \{\mathbf{0}\}$ then the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.
5. Let U, V be 2-dimensional subspaces of $\mathbb{R}^3$. On your first test you showed that the set $U \cap V = \{\mathbf{u} : \mathbf{u} \in U, \mathbf{u} \in V\}$ of vectors in both U and V is a subspace of $\mathbb{R}^3$. Prove that $\dim(U \cap V) \neq 0$. (Hint: let $\{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis for U and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for V. What can you say about $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$?
6. Suppose $S, T : V \to V$ are linear and have the property that $S(T(\mathbf{v})) = T(S(\mathbf{v}))$ for every $\mathbf{v} \in V$. If $\mathbf{v}$ is an eigenvector of T, prove that $S(\mathbf{v})$ is also an eigenvector of T.
7. Suppose $L : V \to V$ is a linear transformation of rank k. Prove that L has at most $k + 1$ distinct eigenvalues.

For each of the following sets, check:

- Does it span the (implicitly given) vector space?
- Is it linearly independent?
- Is it a basis?

1. $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$
2. $S = \{(2, 1, -2), (3, 2, -2), (2, 2, 0)\}$
3. $S = \{(1, 0, 0, 1), (0, 1, 0, 0), (2, 3, 0, 2)\}$
4. $S = \{(1, 5, 2), (3, 1, 4), (-1, 3, 7), (2, 8, 1)\}$

5. $S = \{1 + x^2, 1 + x^3, x - x^2, 5 + x^2 - 4x^3\}$
6. $S = \{1 + 2x, x + 2x^2, x^2 + 2x^3, 2 + x^3\}$

Bases

1. Find a basis for $\mathbb{R}^3$ containing $(-1, 3, 2)$ and $(5, 4, 1)$.
2. Find a basis for $\mathbb{R}^3$ containing $(7, 1, -3)$ and $(1, 1, 1)$.
3. Find a basis for $\mathbb{R}^4$ containing $(1, 2, 3, 4), (1, 1, 1, 1)$, and $(0, 0, 1, 1)$.
4. Find a basis for $\mathcal{P}_3(x)$ containing $1 + 3x^3, x^2 - x, 6 - 2x$.
5. Find a basis for $\mathbb{R}^3$ that is a subset of $\{(1, 1, 1), (2, 4, 6), (7, -1, 2), (2, 5, -2), (3, -6, 4)\}$.
6. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(1, 3), (2, 4), (1, 1)\}$.
7. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(-1, 4), (7, -2), (3, 6)\}$.
8. Find a basis for $\mathcal{P}_2(x)$ that is a subset of $\{1 + x, 3 + x^2, 4 + 3x + 2x^2, x^2 - 7x\}$.

For each of the following matrices, find:

(a) The reduced row echelon form.

(b) A basis for the rowspace.

(c) A basis for the columnspace.

(d) The rank.

(e) A basis for the nullspace.

(f) The nullity.

1. $\begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 4 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 2 & 3 & 1 \\ 1 & 5 & 2 & -2 \\ 4 & -4 & 1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} -1 & 3 & 4 \\ 2 & 5 & 2 \\ 0 & 1 & 3 \\ 4 & 1 & -2 \end{bmatrix}$

Find the rank and nullity of the following matrices

You shouldn't need to do any actual computations here. (Hint: Rank-Nullity theorem).

1. $\begin{bmatrix} 1 & 1 & 1 & 32 & 217 & 53-e & 3^3 \\ 0 & 1 & 0 & 512 & 256 & 128 & 64 \\ 0 & 1 & 1 & 12345 & 4^{4^4} & 2 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ -1 & -2 & -3 & -4 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Find the inverses of the following matrices, or show they are not invertible.

1. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \\ 1 & -4 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 2 \\ 7 & 1 & 5 \end{bmatrix}$

For each of the following functions

(a) Identify the domain and codomain.

(b) Determine whether it is a linear transformation.

(c) Prove your answer from part (b).

(d) If it is, find a matrix with respect to the standard basis of $\mathbb{R}^n$.

(e) If it is linear, find the kernel and image.

1. $f(x, y, z) = (3x^2, x + y, 2z - y)$.
2. $f(x, y, z) = (5x + y, z - 3x, y + z)$.
3. $f(x, y, z) = (x + y, z + y)$.
4. $f(x, y) = (x + y, x - y, 1)$.

For each of the following functions

(a) Determine whether it is a linear transformation.

(b) Prove your answer from part (a).

(c) If it is, find a matrix with respect to the given bases.

(d) If it is linear, find the kernel and image.

1. $L : \mathbb{R}^2 \to \mathbb{R}^3$ given by $L(x, y) = (x, y, x + y)$, with respect to $E = \{(1, 1), (1, -1)$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

2. $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by $L(x, y, z) = (x + y + z, x - y)$, with respect to $E = \{(1, 1, 1), (1, -1, 0), (0, 0, 1)\}$ and $F = \{(3, 0), (0, 2)\}$.

3. $L : \mathcal{P}_3(x) \to \mathbb{R}^2$ given by $L(f(x)) = (f(1), f(2))$, with $E = \{1, x, x^2, x^3\}$ and $F = \{(1, 0), (0, 1)\}$.

4. $L : \mathcal{P}_2(x) \to \mathcal{P}_3(x)$ given by $L(f(x)) = \int_0^x f(t)\,dt$, with $E = \{1, x, x^2\}$ and $F = \{1, x, x^2, x^3\}$.

5. $L : \mathbb{R}^3 \to \mathcal{P}_3(x)$ given by $L(a, b, c) = (a + b + c) + ax + bx^2 + cx^3$, with $E = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ and $F = \{1, 1 + x, 1 + x^2, 1 + x^3\}$.

6. The function $R : \mathbb{R}^3 \to \mathbb{R}^3$ given by rotating 90 degrees counterclockwise around the z axis, and then 135 degrees counterclockwize around the x axis.

Isomorphisms

1. Let $f(x, y, z) = (2x - y, y + z, z + x)$ and $g(a, b, c) = (a + b - c, a + 2b - 2c, -a - b + 2c)$. Prove that g is the inverse of f.

2. Let $L(x, y, z) = (x - y, 3x + z, y - 2z)$. Find a formula for L^{-1}. (Do *not* leave your answer as a matrix).

3. Let $T : \mathbb{R}^3 \to \mathcal{P}_2(x)$ be given by $T(a, b, c) = (a - c) + (b - c)x + (a + b + c)x^2$. Find T^{-1}. (Do *not* leave your answer as a matrix).

4. ($\star$) If $f(x) \in \mathcal{P}_2(x)$ such that $f(1) = 4, f(3) = 7, f(4) = 1$, find $f(x)$. (Hint: define an evaluation map from $\mathcal{P}_2(x)$ to $\mathbb{R}^3$).

5. (a) Suppose $L : \mathbb{R}^3 \to \mathcal{P}_2(x)$ is linear and surjective. Prove it is an isomorphism.

   (b) Suppose $T : \mathcal{P}_5(x) \to \mathbb{R}^6$ is linear with trivial kernel. Prove it is an isomorphism.

Eigenvalues and Eigenvectors

Find the characteristic polynomials, eigenvalues (with algebraic multiplicity), and bases for the eigenspaces, of the following matrices.

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Determinants

1. Find all values of k for which $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$ is invertible.

2. Compute the determinants of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix}$$