Math 214 Test 2
Practice Problem Solutions

Instructor: Jay Daigle

This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I expect the test to be five pages covering a subset of this material.

Proofs

1. Let $L : U \to V$ be a linear transformation of vector spaces. Prove that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.

   Solution: Suppose L is one-to-one, and suppose $\mathbf{u} \in \ker(L)$, that is, $L(\mathbf{u}) = \mathbf{0}$. We know that $L(\mathbf{0}) = \mathbf{0}$, so by definition of one-to-one we know that $\mathbf{u} = \mathbf{0}$. Thus $\ker(L) = \{\mathbf{0}\}$.

   Conversely, suppose $\ker(L) = \{0\}$, and supppose $\mathbf{u}, \mathbf{v} \in U$ with $L(\mathbf{u}) = L(\mathbf{v})$. Then by linearity we have

   $$\mathbf{0} = L(\mathbf{u}) - L(\mathbf{v}) = L(\mathbf{u} - \mathbf{v})$$

   so $\mathbf{u} - \mathbf{v} \in \ker(U) = \{\mathbf{0}\}$. Thus $\mathbf{u} - \mathbf{v} = \mathbf{0}$ and so $\mathbf{u} = \mathbf{v}$.

2. Let $L : U \to V$ be a linear transformation.

   (a) If $\dim U > \dim V$, prove that L is not injective. (There is "too much" in U to fit it all in V without repeating).

   (b) If $\dim U < \dim V$, prove that L is not surjective. (There is "not enough" in U to cover all of V).

   (c) Find counterexamples to the converses of these statements. That is, find a function $L : U \to V$ where L is not injective, but $\dim U < \dim V$. And find a function $L : U \to V$ where L is not surjective, but $\dim U > \dim V$.

   Solution:

   (a) We know that $L(U) \subseteq V$ so $\dim L(U) \leq \dim V$. By the Rank-Nullity Theorem we know that $\dim U = \dim L(U) + \dim \ker(L)$, which we can rewrite as $\dim \ker(L) = \dim U + \dim L(U) \geq \dim(U) - \dim(V) > 0$. Thus the kernel is nontrivial, so the function is not injective.

   (b) By the Rank-Nullity Theorem, we know that $\dim U = \dim \ker(L) + \dim L(U) \geq \dim L(U)$. Thus $\dim L(U) \leq \dim(U) < \dim V$ so $L(U) \neq V$.

   (c) Let $L : \mathbb{R} \to \mathbb{R}^3$ be given by $L(x) = (0, 0, 0)$. Then L is not injective, but clearly $\dim \mathbb{R} < \dim \mathbb{R}^3$. Similarly, $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x, y) = 0$. This is not surjective, but $\dim \mathbb{R}^2 > \dim \mathbb{R}$.

3. Let $L : \mathbb{R}^4 \to \mathbb{R}^2$ such that $\ker(L) = \{(x, y, z, w) : x = y, z = w\}$. Prove that the image of L is $\mathbb{R}^2$.

   Solution: Let A be the matrix of L; then $A \in M_2 \times 4$. We see that the kernel of L is $\{\alpha, \alpha, \beta, \beta\} = \{\alpha(1, 1, 0, 0) + \beta(0, 0, 1, 1)\}$ and thus two-dimensional. Thus the nullity of A is 2, and by the rank-nullity theorem, the rank of A is $4 - 2 = 2$.

   Thus the columnspace of A is two-dimensional. But the columnspace of A is the image of L, so the image of L is a two-dimensional subspace of $\mathbb{R}^2$. The only such subspace is $\mathbb{R}^2$ itself.

4. Let $A \in M_{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that if $N(A) = \{\mathbf{0}\}$ then the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

   Solution: There are two approaches.

   One is to recall the theorem from class: let's assume the equation $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}_0$. Then the set of solutions looks like $\{\mathbf{x}_0 + \mathbf{u} : \mathbf{u} \in N(A)\}$. But since $N(A) = \{\mathbf{0}\}$ then the set of solutions is $\{\mathbf{x}_0 + \mathbf{0}\} = \{\mathbf{x}_0\}$ and thus there is only one solution.

   If we don't remember that theorem, we can prove it directly. Suppose $A\mathbf{x} = \mathbf{b} = A\mathbf{y}$. Then $A\mathbf{x} - A\mathbf{y} = 0$, so $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, and thus $\mathbf{x} - \mathbf{y} \in N(A)$.

   But $N(A) = \{\mathbf{0}\}$ so this implies that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, and thus $\mathbf{x} = \mathbf{y}$. Thus any two solutions are equal, so there is at most one solution.

5. Let U, V be 2-dimensional subspaces of $\mathbb{R}^3$. On your first test you showed that the set $U \cap V = \{\mathbf{u} : \mathbf{u} \in U, \mathbf{u} \in V\}$ of vectors in both U and V is a subspace of $\mathbb{R}^3$. Prove that $\dim(U \cap V) \neq 0$. (Hint: let $\{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis for U and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for V. What can you say about $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$?

   Solution: The set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$ is a four-element subset of $\mathbb{R}^3$, and thus is linearly dependent. Thus there are some a_i not all zero such that

   $$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{v}_1 + a_4\mathbf{v}_2 = \mathbf{0}.$$

   If $a_1 = a_2 = 0$ then $a_3\mathbf{v}_1 + a_4\mathbf{v}_2 = \mathbf{0}$, and since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis it is linearly independent, so $a_3 = a_4 = 0$, a contradiction. So we can assume without loss of generality that $a_1 \neq 0$.

   Then we have $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 = -a_3\mathbf{v}_1 - a_4\mathbf{v}_2$. The left-hand side is an element of U, and is non-zero since $a_1 \neq 0$ and $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent. The right-hand side is an element of V. Thus there is some non-zero element of U that is also an element of V, so $U \cap V \neq \{\mathbf{0}\}$.

6. Suppose $S, T : V \to V$ are linear and have the property that $S(T(\mathbf{v})) = T(S(\mathbf{v}))$ for every $\mathbf{v} \in V$. If $\mathbf{v}$ is an eigenvector of T, prove that $S(\mathbf{v})$ is also an eigenvector of T.

   Solution: $T(S(\mathbf{v})) = S(T(\mathbf{v})) = S(\lambda\mathbf{v}) = \lambda S(\mathbf{v})$ so $S(\mathbf{v})$ is an eigenvector of T by definition.

7. Suppose $L : V \to V$ is a linear transformation of rank k. Prove that L has at most $k+1$ distinct eigenvalues.

   Solution: Suppose that L has $k+2$ distinct eigenvalues; then it has at least $k+1$ distinct non-zero eigenvalues, which we will denote $\lambda_1, \ldots, \lambda_{k+1}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ be corresponding eigenvectors.

   Since these vectors have distinct eigenvectors, they are linearly independent. Further, each of them is in the image of L, since $L(\mathbf{v}_i/\lambda_i) = \mathbf{v}_i$. Thus the image of L contains $k+1$ linearly independent vectors, and so has dimension at least $k+1$, which contradicts the assumption that the rank of L is k.

For each of the following sets, check:

- Does it span the (implicitly given) vector space?
- Is it linearly independent?
- Is it a basis?

1. $S = \{(2, 1, -2), (-2, -1, 2), (4, 2, -4)\}$
2. $S = \{(2, 1, -2), (3, 2, -2), (2, 2, 0)\}$
3. $S = \{(1, 0, 0, 1), (0, 1, 0, 0), (2, 3, 0, 2)\}$
4. $S = \{(1, 5, 2), (3, 1, 4), (-1, 3, 7), (2, 8, 1)\}$
5. $S = \{1 + x^2, 1 + x^3, x - x^2, 5 + x^2 - 4x^3\}$
6. $S = \{1 + 2x, x + 2x^2, x^2 + 2x^3, 2 + x^3\}$

Solution:

1. No, no, no.
2.
3. No, no, no.
4.
5.
6. Yes, yes, yes.

Bases

1. Find a basis for $\mathbb{R}^3$ containing $(-1, 3, 2)$ and $(5, 4, 1)$.

   Solution: It looks like $(1, 0, 0)$ is not in the span of $\{(-1, 3, 2), (5, 4, 1)\}$, so we test:

   $$\begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}-1\\3\\2\end{bmatrix} + b\begin{bmatrix}5\\4\\1\end{bmatrix} = \begin{bmatrix}5b-a\\3a+4b\\2a+b\end{bmatrix}$$

   giving

   $$5b - a = 1 \qquad 3a + 4b = 0 \qquad 2a + b = 0.$$

   This gives us $b = -2a$, so $3a - 8a = 0$ implying $a = 0$ and $b = 0$, and then we have $0 = 1$ a contradiction, so $(1, 0, 0)$ is indeed not in the span. Thus basis padding tells us that $\{(-1, 3, 2), (5, 4, 1), (1, 0, 0\}$ is a basis.

2. Find a basis for $\mathbb{R}^3$ containing $(7, 1, -3)$ and $(1, 1, 1)$.

   Solution: It looks like $(1, 0, 0)$ is not in the span of $\{(7, 1, -3), (1, 1, 1)\}$, so we test:

   $$\begin{bmatrix}1\\0\\0\end{bmatrix} = a\begin{bmatrix}7\\1\\-3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}7a+b\\a+b\\b-3a\end{bmatrix}$$

   giving

   $$7a + b = 1 \qquad a + b = 0 \qquad b - 3a = 0.$$

   This gives us $a = -b$ so $b + 3b = 0$ implies $b = 0$ and thus $a = 0$, so we have $0 = 1$, a contradiction. Thus $(1, 0, 0)$ is not in the span, so by basis padding $\{(7, 1, -3), (1, 1, 1), (1, 0, 0)\}$ is a basis for $\mathbb{R}^3$.

3. Find a basis for $\mathbb{R}^4$ containing $(1, 2, 3, 4), (1, 1, 1, 1)$, and $(0, 0, 1, 1)$.

4. Find a basis for $\mathcal{P}_3(x)$ containing $1 + 3x^3, x^2 - x, 6 - 2x$.

   Solution: We guess that you can't get 1 in the span of this set. We check by supposing

   $$1 = a(1 + 3x^3) + b(x^2 - x) + c(6 - 2x) = a + 6c - (b + 2c)x + bx^2 + 3ax^3)$$

   and thus we get the equations

   $$1 = a + 6c \qquad 0 = -b - 2c0 \qquad = b0 \qquad = 3a.$$

   this gives that $0 = a, 0 = b$ from the last two equations, and thus $0 = c$ from the second equation, which gives a contradiction $1 = 0$ from the first equation. Thus 1 is indeed not in the span.

   Thus $\{1, 1 + 3x^2, x^2 - x, 6 - 2x\}$ is a basis for $\mathcal{P}_3(x)$, since it is linearly independent and the dimension of $\mathcal{P}_3(x)$ is 4.

5. Find a basis for $\mathbb{R}^3$ that is a subset of $\{(1,1,1),(2,4,6),(7,-1,2),(2,5,-2),(3,-6,4)\}$.
6. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(1,3),(2,4),(1,1)\}$.
7. Find a basis for $\mathbb{R}^2$ that is a subset of $\{(-1,4),(7,-2),(3,6)\}$.
8. Find a basis for $\mathcal{P}_2(x)$ that is a subset of $\{1+x, 3+x^2, 4+3x+2x^2, x^2-7x\}$.

For each of the following matrices, find:

(a) The reduced row echelon form.

(b) A basis for the rowspace.

(c) A basis for the columnspace.

(d) The rank.

(e) A basis for the nullspace.

(f) The nullity.

1. $\begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so:

   - The rowspace has basis $\{(1,0,0),(0,1,0),(0,0,1)\}$.
   - The columnspace has basis $\{(3,-1,1),(1,4,1),(2,2,1)\}$.
   - The rank is 3.
   - The nullspace is trivial and so has basis $\{\}$.
   - The nullity is zero.

2. $\begin{bmatrix} -2 & 4 & 1 \\ -5 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{bmatrix}$, so

   - The rowspace has basis $\{(1,0,-1/6),(0,1,1/6)\}$.
   - The columnspace has basis $\{(-2,-5,3),(4,1,3)\}$.
   - The rank is 2.
   - The nullspace is $\{\alpha/6, -\alpha/6, \alpha\}$ and so has basis $\{(1,-1,6)\}$ or $\{1/6,-1/6,1\}$.
   - The nullity is one.

3. $\begin{bmatrix} 6 & 2 & 3 & 1 \\ 1 & 5 & 2 & -2 \\ 4 & -4 & 1 & 3 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & -1/5 \end{bmatrix}$, so

   - The rowspace has basis $\{(1,0,0,2/5),(0,1,0,-2/5),(0,0,1,-1/5)\}$.

- The columnspace has basis $\{(6,1,4),(2,5,-4),(3,2,1)\}$.
- The rank is 3.
- The nullspace is $\{(-2\alpha/5, 2\alpha/5, \alpha/5, \alpha)\}$ and so has basis $\{(-2,2,1,5)\}$ or $\{(-2/5, 2/5, 1/5, 1)\}$.
- The nullity is one.

4. $\begin{bmatrix} -1 & 3 & 4 \\ 2 & 5 & 2 \\ 0 & 1 & 3 \\ 4 & 1 & -2 \end{bmatrix}$

   Solution: The reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so

   - The rowspace has basis $\{(1,0,0),(0,1,0),(0,0,1)\}$.
   - The columnspace has basis $\{(-1,2,0,4),(3,5,1,1),(4,2,3,-2)\}$.
   - The rank is 3.
   - The nullspace is trivial and so has basis $\{\}$.
   - The nullity is zero.

Find the rank and nullity of the following matrices

You shouldn't need to do any actual computations here. (Hint: Rank-Nullity theorem).

1. $\begin{bmatrix} 1 & 1 & 1 & 32 & 217 & 53-e & 3^3 \\ 0 & 1 & 0 & 512 & 256 & 128 & 64 \\ 0 & 1 & 1 & 12345 & 4^{4^4} & 2 & 0 \end{bmatrix}$

   Solution: The rows are linearly independent, so the rank is 3. By the rank-nullity theorem, the nullity is 4.

2. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ -1 & -2 & -3 & -4 \end{bmatrix}$

   Solution: The rows are all scalar multiples of each other, so the rank is 1. By the rank-nullity theorem, the nullity is 3.

3. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 1 & 3 & 2 & 4 \end{bmatrix}$

   Solution: The third row is a combination of the first two, so the rank is 2. By the rank-nullity theorem, the nullity is 2.

4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   Solution: There are three linearly independent rows, since the zero rows don't contribute, so the rank is 3. The nullity is 0.

Find the inverses of the following matrices, or show they are not invertible.

1. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

   Solution: We form the augmented matrix $\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$ and row-reduce to get $\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 3 \end{array}\right]$.

   Thus the inverse is $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

2. $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

   Solution: We notice that the two rows are linearly dependent, so the rank is 1 and the nullity is 1, and the matrix has no inverse.

   Alternatively, we form the augmented matrix $\left[\begin{array}{cc|cc} 3 & 1 & 0 & 1 \\ 6 & 2 & 1 & 0 \end{array}\right]$ and row-reduce to get $\left[\begin{array}{cc|cc} 1 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$.

   Thus the matrix has no inverse.

3. $\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \\ 1 & -4 & 3 \end{bmatrix}$

   Solution: We form the augmented matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 1 & -4 & 3 & 0 & 0 & 1 \end{array}\right]$ and row-reduce to get

   $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/27 & 8/27 & 4/27 \\ 0 & 1 & 0 & 5/27 & 1/54 & -13/54 \\ 0 & 0 & 1 & 7/27 & -2/27 & -1/27 \end{array}\right]$. Thus the inverse is $\begin{bmatrix} -1/27 & 8/27 & 4/27 \\ 5/27 & 1/54 & -13/54 \\ 7/27 & -2/27 & -1/27 \end{bmatrix}$.

4. $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 0 & 2 \\ 7 & 1 & 5 \end{bmatrix}$

   Solution: We may notice that the third row is the sum of the first two, and thus the matrix has rank 2 and nullity 1, so is not invertible.

   Alternatively, we form the augmented matrix $\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 & 1 & 0 \\ 7 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$ and row-reduce to get

   $\left[\begin{array}{ccc|ccc} 1 & 0 & 2/5 & 0 & 1/5 & 0 \\ 0 & 1 & 11/5 & 0 & -7/5 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$. Thus there is no inverse.

For each of the following functions

(a) Identify the domain and codomain.

(b) Determine whether it is a linear transformation.

(c) Prove your answer from part (b).

(d) If it is, find a matrix with respect to the standard basis of $\mathbb{R}^n$.

(e) If it is linear, find the kernel and image.

1. $f(x, y, z) = (3x^2, x + y, 2z - y)$.

   Solution: Domain and codomain are $\mathbb{R}^3$.

   No, because $f(1, 0, 0) = (3, 1, 0)$ but $f(2, 0, 0) = (12, 2, 0) \neq 2(3, 1, 0)$.

2. $f(x, y, z) = (5x + y, z - 3x, y + z)$.

Solution: Domain and codomain are $\mathbb{R}^3$.

$$\begin{aligned} f(r(x, y, z)) &= (5rx + ry, rz - 3rx, ry + rz) = r(5x + y, z - 3x, y + z) = rf(x, y, z) \\ f((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= f(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (5(x_1 + x_2) + (y_1 + y_2), (z_1 + z_2 - 3(x_1 + x_2), (y_1 + y_2) + (z_1 + z_2) \\ &= (5x_1 + y_1, z_1 - 3x_1, y_1 + z_2) + (5x_2 + y_2, z_2 - 3x_2, y_2 + z_2) \\ &= f(x_1, y_1, z_1) + f(x_2, y_2, z_2). \end{aligned}$$

Thus it is linear. We have

$$\begin{aligned} f(\mathbf{e}_1) &= (5, -3, 0) \\ f(\mathbf{e}_2) &= (1, 0, 1) \\ f(\mathbf{e}_3) &= (0, 1, 1) \\ A &= \begin{bmatrix} 5 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

To find the kernel and image, we row reduce the matrix, getting

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which tells us that the kernel is trivial, and the image is $\text{span}\{(5, -3, 0), (1, 0, 1), (0, 1, 1)\}$. Since this is three-dimensional, the image must be $\mathbb{R}^3$.

3. $f(x, y, z) = (x + y, z + y)$.

Solution: The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$.

The function is a linear transformation, since:

$$\begin{aligned} f(r(x, y, z)) &= (rx + ry, rz + ry) = r(x + y, z + y) = rf(x, y, z) \\ f((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= ((x_1 + x_2) + (y_1 + y_2), (z_1 + z_2) + (y_1 + y_2)) \\ &= (x_1 + y_1, z_1 + y_1) + (x_2 + y_2, z_2 + y_2) = f(x_1, y_1, z_2) + f(x_2, y_2, z_2). \end{aligned}$$

We compute

$$\begin{aligned} f(\mathbf{e}_1) &= (1, 0) \\ f(\mathbf{e}_2) &= (1, 1) \\ f(\mathbf{e}_3) &= (0, 1) \\ A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

The row reduced echelon form is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ so the kernel is $x = z, y = -z$, or $\{(\alpha, -\alpha, \alpha)\} = \{\alpha(1, -1, 1)\}$. Thus $\{(1, -1, 1)\}$ is a basis for the kernel.

The image is spanned by $(1, 0)$ and $(1, 1)$ since these are the columns corresponding to leading ones. THus $\{(1, 0), (1, 1)\}$ is a basis for the image, which must be all of $\mathbb{R}^2$.

4. $f(x, y) = (x + y, x - y, 1)$.

Solution: The domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$.

The function is not linear, since $f(0, 0) = (0, 0, 1) \neq \mathbf{0}$. Alternately, we can see that, say, $f(1, 0) = (1, 1, 1)$ and $f(0, 1) = (1, -1, 1)$, and $f(1, 1) = (2, 0, 1) \neq (1, 1, 1) + (1, -1, 1)$.

For each of the following functions

(a) Determine whether it is a linear transformation.

(b) Prove your answer from part (a).

(c) If it is, find a matrix with respect to the given bases.

(d) If it is linear, find the kernel and image.

1. $L : \mathbb{R}^2 \to \mathbb{R}^3$ given by $L(x, y) = (x, y, x + y)$, with respect to $E = \{(1, 1), (1, -1)$ and $F = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.

   Solution: L is a linear transformation, since

$$\begin{aligned} L(r(x,y)) &= (rx, ry, rx+ry) = r(x, y, x+y) = rL(x,y) \\ L((x_1,y_1)+(x_2,y_2)) &= (x_1+x_2, y_1+y_2, x_1+x_2+y_1+y_2) \\ &= (x_1, y_1, x_1+y_1) + (x_2, y_2, x_2+y_2) = L(x_1,y_1) + L(x_2,y_2). \end{aligned}$$

   To find the matrix, we compute

$$\begin{aligned} L(1,1) &= (1,1,2) = -(1,1,0) + 2(1,1,1) \to (0,-1,2) \\ L(1,-1) &= (1,-1,0) = 2(1,0,0) - 1(1,1,0) \to (2,-1,0) \\ A &= \begin{bmatrix} 0 & 2 \\ -1 & -1 \\ 2 & 0 \end{bmatrix}. \end{aligned}$$

   To find the kernel and image, we row-reduce the matrix to get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Thus the kernel is trivial, and the image is the span of $\{(0, -1, 2), (2, -1, 0)\}$ in F; in other words, the span of $\{(1, 1, 2), (1, -1, 0)\}$. (Which is of course the span of the image of the basis, as we proved on your last homework!)

2. $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by $L(x, y, z) = (x + y + z, x - y)$, with respect to $E = \{(1, 1, 1), (1, -1, 0), (0, 0, 1)\}$ and $F = \{(3, 0), (0, 2)\}$.

   Solution: L is a linear transformation, since

$$\begin{aligned} L(r(x,y,z)) &= (rx+ry+rz, rx-ry) = r(x+y+z, x-y) = rL(x,y,z) \\ L((x_1,y_1,z_1)+(x_2,y_2,z_2)) &= (x_1+x_2+y_1+y_2+z_1+z_2, x_1+x_2-(y_1+y_2)) \\ &= (x_1+y_1+z_1, x_1-y_1) + (x_2+y_2+z_2, x_2-y_2) \\ &= L(x_1,y_1,z_1) + L(x_2,y_2,z_2). \end{aligned}$$

   To find the matrix, we compute

$$\begin{aligned} L(1,1,1) &= (3,0) \to (1,0) \\ L(1,-1,0) &= (0,2) \to (0,1) \\ L(0,0,1) &= (1,0) \to (1/3,0) \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}.$$

   To find the kernel, we see the matrix is already row-reduced. Thus $N(A) = \{(-\alpha/3, 0, \alpha)\} = \{\alpha(-1/3, 0, 1)\}$, so a basis for the kernel is $\{(-1, 0, 3)\}$ or $\{-1/3, 0, 1\}$ in E. Thus a basis for the kernel is $\{(-1, -1, 2)\}$.

   The image is the span of $\{(1, 0), (0, 1)\}$ in F, and thus the span of $\{(3, 0), (0, 2)\}$. Thus $\{(3, 0), (0, 2)\}$ is a basis for the image.

3. $L : \mathcal{P}_3(x) \to \mathbb{R}^2$ given by $L(f(x)) = (f(1), f(2))$, with $E = \{1, x, x^2, x^3\}$ and $F = \{(1, 0), (0, 1)\}$.

Solution: This is a linear transformation, since

$$\begin{aligned} L(rf(x)) &= (rf(1), rf(2)) = r(f(1), f(2)) = rL(f(x)) \\ L((f+g(x)) &= ((f+g)(1), (f+g)(2)) = (f(1)+g(1), f(2)+g(2)) \\ &= (f(1), f(2)) + (g(1), g(2)) = L(f(x)) + L(g(x)). \end{aligned}$$

To compute the matrix, we have

$$\begin{aligned} L(1) &= (1, 1) \\ L(x) &= (1, 2) \\ L(x^2) &= (1, 4) \\ L(x^3) &= (1, 8) \\ A &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix}. \end{aligned}$$

Row reducing this matrix gives

$$\begin{bmatrix} 1 & 0 & -2 & -6 \\ 0 & 1 & 3 & 7 \end{bmatrix}$$

so the kernel is given by $\{(2\alpha + 6\beta, -3\alpha - 7\beta, \alpha, \beta)\} = \{\alpha(2, -3, 1, 0) + \beta(6, -7, 0, 1)\}$ in E, and thus a basis is $\{2 - 3x + x^2, 6 - 7x + x^3\}$.

The image has basis $\{(1, 1), (1, 2)\}$ and thus is all of $\mathbb{R}^2$.

4. $L : \mathcal{P}_2(x) \to \mathcal{P}_3(x)$ given by $L(f(x)) = \int_0^x f(t)\,dt$, with $E = \{1, x, x^2\}$ and $F = \{1, x, x^2, x^3\}$.

Solution: L is linear, since from calculus we know that $\int_0^x rf(t)\,dt = r\int_0^x f(t)\,dt$, and $\int_0^x f(t) + g(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$.

To find the matrix we compute

$$\begin{aligned} L(1) &= \int_0^x 1\,dt = x \to (0, 1, 0, 0) \\ L(x) &= \int_0^x t\,dt = x^2/2 \to (0, 0, 1/2, 0) \\ L(x^2) &= \int_0^x t^2\,dt = x^3/3 \to (0, 0, 0, 1/3) \end{aligned}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

The reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the kernel is trivial, and the image is the span of $\{(0, 1, 0, 0), (0, 0, 1/2, 0), (0, 0, 0, 1/3)\}$ in F, or in other words the span of $\{(x, x^2/2, x^3/3)\}$. Thus the image is all cubic polynomials with zero constant term.

5. $L : \mathbb{R}^3 \to \mathcal{P}_3(x)$ given by $L(a,b,c) = (a+b+c) + ax + bx^2 + cx^3$, with $E = \{(1,1,1),(1,1,0),(1,0,0)\}$ and $F = \{1, 1+x, 1+x^2, 1+x^3\}$.

   Solution: The function is linear, since

$$\begin{aligned} L(r(a,b,c)) &= (ra+rb+rc) + rax + rbx^2 + rcx^3 = rL(a,b,c) \\ L((a_1,b_1,c_1)+(a_2,b_2,c_2)) &= (a_1+a_2+b_1+b_2+c_1+c_2) + (a_1+a_2)x + (b_1+b_2)x^2 + (c_1+c_2)x^3 \\ &= (a_1+b_1+c_1) + a_1x + b_1x^2 + c_1x^3 + (a_2+b_2+c_2) + a_2x + b_2x^2 + c_2x^3 \\ &= L(a_1,b_1,c_1) + L(a_2,b_2,c_2). \end{aligned}$$

   To find the matrix we compute

$$\begin{aligned} &L(1,1,1) = 3 + x + x^2 + x^3 \to (0,1,1,1) \\ &L(1,1,0) = 2 + x + x^2 \to (0,1,1,0) \\ &L(1,0,0) = 1 + x \to (0,1,0,0) \\ &A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

   The row-reduced form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   so the kernel is trivial. The image is the span of $\{3 + x + x^2 + x^3, 2 + x + x^2, 1 + x\}$.

6. The function $R : \mathbb{R}^3 \to \mathbb{R}^3$ given by rotating 90 degrees counterclockwise around the z axis, and then 135 degrees counterclockwize around the x axis.

   Solution: This function is linear since rotations are linear: stretching then rotating is the same as rotating then stretching, and adding then rotating is the same as rotating then adding.

   To find the matrix we compute:

$$\begin{aligned} &R(1,0,0) = (0, -\sqrt{2}/2, \sqrt{2}/2) \\ &R(0,1,0) = (-1,0,0) \\ &R(0,0,1) = (0, -\sqrt{2}/2, -\sqrt{2}/2) \\ &A = \begin{bmatrix} 0 & -1 & 0 \\ -\sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{bmatrix}. \end{aligned}$$

   It's not too hard to see that the rows are linearly independent, so the rank is 3 and thus a basis for the columnspace is $\{(0, -\sqrt{2}/2, \sqrt{2}/2), (-1,0,0), (0, -\sqrt{2}/2, -\sqrt{2}/2)\}$, or $\{(0,-1,1),(1,0,0),(0,1,1)\}$. Thus the columnspace spans $\mathbb{R}^3$. (This makes sense since we know we can land anywhere by rotating). Since the rank is 3, the kernel is trivial, which again makes geometric sense.

Isomorphisms

1. Let $f(x,y,z) = (2x - y, y + z, z + x)$ and $g(a,b,c) = (a + b - c, a + 2b - 2c, -a - b + 2c)$. Prove that g is the inverse of f.

Solution:

$$
\begin{aligned}
g(f(x,y,z)) &= g(2x-y, y+z, z+x) \\
&= \begin{bmatrix} (2x-y)+(y+z)-(z+x) \\ (2x-y)+2(y+z)-2(z+x) \\ -(2x-y)-(y+z)+2(z+x) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \\
f(g(a,b,c) &= f(a+b-c, a+2b-2c, -a-b+2c) \\
&= \begin{bmatrix} 2(a+b-c)-(a+2b-2c) \\ (a+2b-2c)+(-a-b+2c) \\ (-a-b+2c)+(a+b-c) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\end{aligned}
$$

2. Let $L(x,y,z) = (x-y, 3x+z, y-2z)$. Find a formula for L^{-1}. (Do *not* leave your answer as a matrix).

   Solution: We have

$$
\begin{aligned}
L(1,0,0) &= (1,3,0) \\
L(0,1,0) &= (-1,0,1) \\
L(0,0,1) &= (0,1,-2) \\
A &= \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}
\end{aligned}
$$

   and we compute

$$
\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 7 & -3 & 1 & -3 \end{array}\right]
$$

$$
\to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3/7 & 1/7 & -3/7 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/7 & 2/7 & 1/7 \\ 0 & 1 & 0 & -6/7 & 2/7 & 1/7 \\ 0 & 0 & 1 & -3/7 & 1/7 & -3/7 \end{array}\right]
$$

   so

$$
A^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 & 1 \\ -6 & 2 & 1 \\ -3 & 1 & -3 \end{bmatrix}.
$$

   Thus

$$
L^{-1}(a,b,c) = \frac{1}{7}(a+2b+c, -6a+2b+c, -3a+b-3c).
$$

3. Let $T : \mathbb{R}^3 \to \mathcal{P}_2(x)$ be given by $T(a,b,c) = (a-c)+(b-c)x+(a+b+c)x^2$. Find T^{-1}. (Do *not* leave your answer as a matrix).

   Solution: We use the standard bases of $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{1,x,x^2\}$. We have

$$
\begin{aligned}
T(1,0,0) &= 1+x^2 \to (1,0,1) \\
T(0,1,0) &= x+x^2 \to (0,1,1) \\
T(0,0,1) &= -1-x+x^2 \to (-1,-1,1) \\
A &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.
\end{aligned}
$$

We compute

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & -1 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & 2/3 & 1/3 \\ 0 & 0 & 1 & -1/3 & -1/3 & 1/3 \end{array}\right]$$

so

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

Thus

$$T^{-1}(a + bx + cx^2) = \frac{1}{3}\begin{bmatrix} 2a - b + c \\ -a + 2b + c \\ -a - b + c \end{bmatrix}.$$

4. ($\star$) If $f(x) \in \mathcal{P}_2(x)$ such that $f(1) = 4, f(3) = 7, f(4) = 1$, find $f(x)$. (Hint: define an evaluation map from $\mathcal{P}_2(x)$ to $\mathbb{R}^3$).

Solution: Define a function $E : \mathcal{P}_2(x) \to \mathbb{R}^3$ by $E(f) = (f(1), f(3), f(4))$. This is linear, and we can compute the matrix:

$$\begin{aligned} E(1) &= (1,1,1) \\ E(x) &= (1,3,4) \\ E(x^2) &= (1,9,16) \\ A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}. \end{aligned}$$

We can invert this matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 9 & 0 & 1 & 0 \\ 1 & 4 & 16 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 8 & -1 & 1 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1/2 & 1/2 & 0 \\ 0 & 3 & 15 & -1 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 3/2 & -1/2 & 0 \\ 0 & 1 & 4 & -1/2 & 1/2 & 0 \\ 0 & 0 & 3 & 1/2 & -3/2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 3/2 & -1/2 & 0 \\ 0 & 1 & 4 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/6 & -1/2 & 1/3 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -7/6 & 5/2 & -4/3 \\ 0 & 0 & 1 & 1/6 & -1/2 & 1/3 \end{array}\right]$$

so

$$A^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ -7/6 & 5/2 & -4/3 \\ 1/6 & -1/2 & 1/3 \end{bmatrix}.$$

Now we just need to compute

$$\begin{aligned} A^{-1}(4,7,1) &= \begin{bmatrix} 2 & -2 & 1 \\ -7/6 & 5/2 & -4/3 \\ 1/6 & -1/2 & 1/3 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 23/2 \\ -5/2 \end{bmatrix} \\ f(x) &= -5 + \frac{23}{2}x - \frac{5}{2}x^2. \end{aligned}$$

5. (a) Suppose $L : \mathbb{R}^3 \to \mathcal{P}_2(x)$ is linear and surjective. Prove it is an isomorphism.
   (b) Suppose $T : \mathcal{P}_5(x) \to \mathbb{R}^6$ is linear with trivial kernel. Prove it is an isomorphism.

   Solution:

   (a) We just need to prove that L is injective, that is, that it has trivial kernel. By rank-nullity, we know that $\dim \mathbb{R}^3 = \dim \mathcal{P}_2(x) + \dim \ker(L)$. But $\dim \mathbb{R}^3 = 3 = \dim \mathcal{P}_2(x)$, so we have $3 = 3 + \dim \ker(L)$, so $\dim \ker(L) = 0$ and the kernel is trivial.
   Thus L is injective, and since it is also surjective, it is an isomorphism.

   (b) T has trivial kernel, so it is injective. We just need to prove that it is surjective.
   We know that $\dim \mathcal{P}_5(x) = 6 = \dim \mathbb{R}^6$, and by rank-nullity theorem we have $\dim \mathcal{P}_5(x) = \dim T(\mathcal{P}_5(x)) + \dim \ker(T)$. But the kernel is trivial, so $\dim \ker(T) = 0$, and thus we have $6 = \dim T(\mathcal{P}_5(x))$. Thus $T(\mathcal{P}_5(x))$ is a six-dimensional subspace of a six-dimensional space, and hence $T(\mathcal{P}_5(x)) = \mathbb{R}^6$. So T is surjective, and thus an isomorphism.

Eigenvalues and Eigenvectors

Find the characteristic polynomials, eigenvalues (with algebraic multiplicity), and bases for the eigenspaces, of the following matrices.

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

   Solution: $\chi(\lambda) = -\lambda^3 + 2\lambda^2 - \lambda - 2$ has roots $2, 1, -1$. The corresponding eigenvectors are $(1,1,1), (-1,-1,2), (-1,1,0)$.

2. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

   Solution: $\chi(\lambda) = \lambda^2 - \lambda^3$ has roots $1, 0, 0$ with corresponding eigenvectors $(1,0,1)$ and $(2,-1,1)$.

3. $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

   Solution: $\chi(\lambda) = -\lambda^3 + 4\lambda^2 - 4\lambda$ has roots $2, 2, 0$, with corresponding eigenvectors $(-1,0,1), (-1,1,0), (1,0,1)$.

4. $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

   Solution: $\chi(\lambda) = -\lambda^3 + 9\lambda^2 - 27\lambda + 27$ has roots $3, 3, 3$, with corresponding eigenvectors $(-1,0,1)$ and $(0,1,0)$.

5. $\begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

   Solution: $\chi(\lambda) = \lambda^4 - 9\lambda^3 + 18\lambda^2 + 32\lambda - 96$ has roots $4, 4, 3, -2$ with corresponding eigenvectors $(0,1,0,0), (1,0,0,0), (-1,-2,1,1), (2,-1,-12,18)$.

6. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

   Solution: $\chi(\lambda) = \lambda^4 - 8\lambda^3 + 23\lambda^2 - 28\lambda + 12$ has roots $3, 1, 2, 2$ with corresponding eigenvalues $(1,2,0,0), (0,1,0,0), (0,0,-1,0)$. Note that $\dim E_2 = 1$.

Determinants

1. Find all values of k for which $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$ is invertible.

   Solution: We compute $\det(A) = k(k-2)^2$. We know the matrix is invertible if and only if $\det A \neq 0$, so A is invertible unless $k = 0$ or $k = 2$.

2. Compute the determinants of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix}$$

   Solution:

$$\begin{array}{ccc} 0 & -2 & -12 \\ 4 & 8 & 0. \end{array}$$

   In particular notice that in the fourth and fifth matrices, we can pick the third row and second column rspectively to make our job much easier; in the sixth matrix, the third row is the sum of the first two so the matrix must be linearly dependent.