Math 214 Final Exam
Practice Problem Solutions

Instructor: Jay Daigle

This is not a practice test, in the sense that it is not the format I expect the test to be. It is a collection of practice problems. I will update you when I finalize the test format.

Proofs

1. Let Q be the subspace of $\mathcal{P}(x)$ consisting of polynomials with zero constant term. Prove that the function $D : Q \to \mathcal{P}(x)$ given by the derivative is an isomorphism.

   Solution: We know that D is linear, so we just need to prove that it is one-to-one and onto. Suppose $D(a_1x + \cdots + a_nx^n) = 0$. then we have $0 = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$ and thus $a_1 = 2a_2 = \cdots = na_n = 0$ so $a_1 = a_2 = \cdots = a_n = 0$. Thus $D(f) = 0$ implies $f = 0$, so $\ker(D) = \{0\}$ and thus D is one-to-one.

   Conversely, let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathcal{P}(x)$. Then let $g(x) = a_0x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1} \in Q$, and we see that $D(g) = f$. Thus D is onto.

   Consequently we see that D is one-to-one and onto, thus it is an isomorphism by definition.

2. Let $U = \operatorname{span}\{x, \sin(x), \cos(x), x^5, 1\}$. Find an isomorphism between U and $\mathbb{R}^5$.

   Solution: Define L by

   $$L(a_1x + a_2\sin(x) + a_3\cos(x) + a_4x^5 + a_5) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}.$$

   Then L takes a basis to a basis, and thus is an isomorphism.

3. Suppose V is a vector space and $L : V \to \mathbb{R}^5$ is surjective and $\dim\ker(L) = 2$. What can you say about V?

   Solution: By the rank-nullity theorem, $\dim V = \dim\ker(L) + \dim L(V)$. We know that $L(V) = \mathbb{R}^5$ so $\dim L(V) = 5$, and $\dim\ker(L) = 2$. Thus $\dim V = 7$.

4. Suppose $T : \mathbb{R}^5 \to \mathcal{P}_4(x)$ and $\dim\ker(T) = 1$. What can you say about $T(\mathbb{R}^5)$?

   Solution: By the Rank-Nullity Theorem, we know that $\dim\mathbb{R}^5 = \dim\ker(T) + \dim T(\mathbb{R}^5)$, and thus $5 = 1 + \dim T(\mathbb{R}^5)$, so $\dim T(\mathbb{R}^5)$ is four-dimensional. Thus T is not surjective since $\mathcal{P}_4(x)$ is five-dimensional.

5. If λ is an eigenvalue of A then prove that λ^{-1} is an eigenvalue of A^{-1}.

   Solution: Let $\mathbf{v} \in E_\lambda$ be an eigenvector with eigenvalue λ. Then $A\mathbf{v} = \lambda\mathbf{v}$, which implies that $A^{-1}(\lambda\mathbf{v})A^{-1}A\mathbf{v} = \mathbf{v}$. Dividing both sides by λ, we have $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

Things to Ponder

1. Find a 4×4 matrix with no real eigenvalues. Is it possible to find a 3×3 matrix with no real eigenvalues?

   Solution: We want to find a matrix whose characteristic polynomial has no real roots. The simplest and most obvious such polynomial is $(x^2+1)^2$, so we want to build one of these. The simplest way to do *that* is to find a 2×2 matrix with characteristic polynomial x^2+1 and repeat it twice.

   We've actaully seen this matrix before; it's the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which has characteristic polynomial

   $$\chi(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = (-\lambda)^2 - (-1\cdot 1) = \lambda^2+1.$$

   To get a 4×4 matrix we can glue two copies of this together. We set

   $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

   which you can see has characteristic polynomial $\chi_A(\lambda) = (\lambda^2+1)(\lambda^2+1)$. This has no real roots, so the matrix has no real eigenvalues.

   For a 3×3 matrix, we would be looking for a degree 3 polynomial with no real roots. No such polynomial exists, so every 3×3 matrix has a real eigenvalue.

   (For similar reasons, it is a theorem that every matrix has a *complex* eigenvalue).

2. Find matrices $A, B \in M_{n\times n}$ such that $\operatorname{Tr}(A)\operatorname{Tr}(B) \neq \operatorname{Tr}(AB)$.

   Find a matrix A such that $\operatorname{Tr}(A^2) < 0$.

   Solution:

   Solving the second will also solve the first.

   Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $\operatorname{Tr}(A) = 0$. But

   $$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \operatorname{Tr}(A^2) = -2 \neq 0^2.$$

   Secretly what's going on here is that A^2 has the eigenvalues $\pm i$, so A^2 has the eigenvalues $(\pm i)^2$, both of which are -1.

3. What happens if you use the Gram-Schmidt process on a set of vectors that isn't linearly independent?

   Solution: When you get to the vector that is a linear combination of the previous vectors, it will equal the sum of its projections onto them. So one of your vectors will be transformed into zero.

   That is, if $\mathbf{e}_3 \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, then $\mathbf{f}_3 = \mathbf{e}_3 - \operatorname{proj}_{\mathbf{e}_1}\mathbf{e}_3 - \operatorname{proj}_{\mathbf{e}_2}\mathbf{e}_3 = 0$.

   Thus the Gram-Schmidt process can be used to turn a spanning set into a basis, by throwing out the vectors that become zero.

Find the transition matrices between the following bases

1. The standard basis and

   $$F = \left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \right\}$$

Solution: The transition matrix from F to the standard basis is

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & 6 \\ 1 & 4 & 3 \end{bmatrix}.$$

The transition matrix from the standard basis to F is

$$A^{-1} = \begin{bmatrix} 3/14 & 1/35 & -9/70 \\ 0 & -1/5 & 2/5 \\ -1/14 & 9/35 & -11/70 \end{bmatrix}.$$

2. The standard basis and

$$F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Solution: The transition matrix from F to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The transition matrix from the standard basis to F is

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}.$$

3.

$$E = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution:

The transition matrix from E to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The transition matrix from F to the standard basis is

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and the transition matrix from the standard basis to F is

$$B^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

So the transition matrix from E to F is

$$B^{-1}A = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1 \\ 1 & 1/2 & 0 \\ 1 & 1/2 & 1 \end{bmatrix}$$

and the transition matrix from F to E is

$$(B^{-1}A)^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}.$$

4.

$$E = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution: The transition matrix from E to the standard basis is

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transition matrix from the standard basis to F is then

$$B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the transition matrix from E to F is

$$B^{-1}A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to E is

$$(B^{-1}A)^{-1} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 2 & 1/2 \\ -3/2 & -5 & -1/2 \end{bmatrix}.$$

Write the given element in the given basis

1. Write $(3, 1, 4)$ in the basis $F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   Solution: The transition matrix from F to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so the transition matrix from the standard basis to F is the inverse inverse

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$[(3, 1, 4)]_F = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

2. Write $(2,7,1)$ in the basis $F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   Solution: The transition matrix from F to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   so the transition matrix from the standard basis to F is

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

   so

$$[(2,7,1)]_F = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}.$$

3. Write $(1,-1,0)$ in the basis $F = \left\{ \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   Solution: The transition matrix from F to the standard matrix is

$$A = \begin{bmatrix} 3 & 7 & 1 \\ 5 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

   so the transition matrix from the standard basis to F is

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 3 & 3 & -6 \\ 3 & -1 & -2 \\ -18 & -2 & 24 \end{bmatrix}$$

   and

$$[(1,-1,0)]_F = \frac{1}{12} \begin{bmatrix} 3 & 3 & -6 \\ 3 & -1 & -2 \\ -18 & -2 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ -4/3 \end{bmatrix}.$$

4. Write $(2,3,4)$ in the basis $F = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

   Solution:

   The transition matrix from F to the standard basis is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

   so the transition matrix from the standard basis to F is

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

   and

$$[(2,3,4)]_F = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}.$$

Find the matrix of the operator with respect to the given basis

1. Give the matrix of $L(x, y, z) = (3x+y+z, 5x-2y+z, y+z)$ with respect to $F = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\}$.

   Solution:

   The matrix of L with respect to the standard basis is

   $$A = \begin{bmatrix}3 & 1 & 1\\5 & -2 & 1\\0 & 1 & 1\end{bmatrix}.$$

   The transition matrix from F to the standard basis is

   $$S = \begin{bmatrix}1 & 1 & 1\\1 & 0 & 0\\0 & 1 & -1\end{bmatrix}$$

   with inverse

   $$S^{-1} = \begin{bmatrix}0 & 1 & 0\\1/2 & -1/2 & 1/2\\1/2 & -1/2 & -1/2\end{bmatrix}.$$

   Thus the matrix of L with respect to F is

   $$S^{-1}AS = \begin{bmatrix}3 & 6 & 4\\1 & -1/2 & -3/2\\0 & -3/2 & -1/2\end{bmatrix}.$$

2. Give the matrix of $L(x, y, z) = (2x+3y-z, 4x-y+3z, 2x+z)$ with respect to $F = \left\{ \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix} \right\}$.

   Solution:

   The matrix of L with respect to the standard basis is

   $$A = \begin{bmatrix}2 & 3 & -1\\4 & -1 & 3\\2 & 0 & 1\end{bmatrix}$$

   and the transition matrix from F to the standard basis is

   $$S = \begin{bmatrix}0 & 1 & 1\\1 & 2 & 1\\0 & 1 & -1\end{bmatrix}$$

   with

   $$S^{-1} = \begin{bmatrix}-3/2 & 1 & -1/2\\1/2 & 0 & 1/2\\1/2 & 0 & -1/2\end{bmatrix}.$$

   Thus the matrix of L with respect to F is

   $$S^{-1}AS = \begin{bmatrix}-11/2 & -7 & -19/2\\3/2 & 5 & 7/2\\3/2 & 2 & 5/2\end{bmatrix}.$$

3. Give the matrix of $L(x, y, z) = (-x+4y+2z, 3x-5y+2, 3x+2y)$ with respect to $F = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution: The matrix of L with respect to the standard basis is

$$A = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -5 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

with inverse

$$S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Thus the matrix of L with respect to F is

$$S^{-1}AS = \begin{bmatrix} 0 & 5 & 3 \\ 5 & -2 & 0 \\ 0 & -2 & -4 \end{bmatrix}.$$

4. Give the matrix of $L(x, y, z) = (2x-y, 3x+y+4z, x+2y+z)$ with respect to $F = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution:

The matrix of L with respect to the standard basis is

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

and the transition matrix from F to the standard basis is

$$S = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

with inverse

$$S^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}.$$

Thus the matrix of L with respect to F is

$$S^{-1}AS = \begin{bmatrix} 7 & 4 & 2 \\ 1 & 0 & -1 \\ -18 & -11 & 3 \end{bmatrix}.$$

Angles and Magnitudes

1. Compute

$$\begin{bmatrix}3\\1\\2\end{bmatrix}\cdot\begin{bmatrix}5\\7\\-1\end{bmatrix},\quad \begin{bmatrix}4\\1\\3\\5\end{bmatrix}\cdot\begin{bmatrix}2\\-5\\7\\4\end{bmatrix},\quad \begin{bmatrix}1\\3\\2\end{bmatrix}\cdot\begin{bmatrix}4\\-1\\-1\end{bmatrix},\quad \begin{bmatrix}7\\1\\5\end{bmatrix}\cdot\begin{bmatrix}-3\\1\\1\end{bmatrix}.$$

Solution: $20, 44, -1, -15.$

2. Find the magnitudes and corresponding unit vectors for

$$\begin{bmatrix}3\\1\\2\end{bmatrix},\quad \begin{bmatrix}5\\12\end{bmatrix},\quad \begin{bmatrix}4\\2\\-2\end{bmatrix},\quad \begin{bmatrix}7\\-1\\-3\end{bmatrix}.$$

Solution: $\sqrt{9+1+4} = \sqrt{13}, \sqrt{25+144} = 13, \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}, \sqrt{49+1+9} = \sqrt{59}.$

3. Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ for

(a) $\mathbf{u} = (5, 2), \mathbf{v} = (-3, 4)$

(b) $\mathbf{u} = (2, 1), \mathbf{v} = (7, 1)$

(c) $\mathbf{u} = (3, 1, 4), \mathbf{v} = (2, 1, 1)$

(d) $\mathbf{u} = (2, 1, 1), \mathbf{v} = (-4, -1, -1)$

(e) $\mathbf{u} = (5, 0, 0), \mathbf{v} = (3, 2, 1)$.

Solution:

(a) $\frac{-7}{25}\begin{bmatrix}-3\\4\end{bmatrix}$

(b) $\frac{15}{50}\begin{bmatrix}7\\1\end{bmatrix}$

(c) $\frac{11}{6}\begin{bmatrix}2\\1\\1\end{bmatrix}$

(d) $\frac{-10}{18}\begin{bmatrix}-4\\-1\\-1\end{bmatrix}$

(e) $\frac{15}{14}\begin{bmatrix}3\\2\\1\end{bmatrix}$

Diagonalization Theory

1. In class we saw that

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}-1&3&-1\\1&-2&1\\1&-3&2\end{bmatrix}\begin{bmatrix}2&-3&1\\1&-2&1\\1&-3&2\end{bmatrix}\begin{bmatrix}1&3&-1\\1&1&0\\1&0&1\end{bmatrix}.$$

Multiply out the three matrices on the right and confirm that this works.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What are the eigenvalues of A? Is $A^2 = A$? Why not?

   Solution: $\chi_A(\lambda) = (1-\lambda)^2$ has roots $1, 1$, so the eigenvalues are 1. We compute that

   $$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \neq A.$$

   In class we argued that if a diagonalizable matrix has eigenvalues all equal to 1 and 0, then $A^n = A$. This matrix has all eigenvalues 1, but it is not in fact diagonalizable since $\dim E_1 = 1$. Thus the same principle does not hold.

3. Show the following pairs of matrices are not similar:

   $$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix}$$

   $$C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 1 & 3 \end{bmatrix}$$

   $$E = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 8 & -2 \\ 0 & 0 & 10 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 0 \\ 5 & 3 & 12 \end{bmatrix}$$

   $$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Solution: $\operatorname{Tr}(A) = 5$ and $\operatorname{Tr}(B) = 2$ so the matrices aren't similar.

   $\operatorname{Tr}(C) = \operatorname{Tr}(D) = 8$, but $\det(C) = 16$ and $\det(D) = 18$ so the matrices aren't similar.

   $\operatorname{Tr}(E) = \operatorname{Tr}(F) = 21$ and $\det(E) = \det(F) = 240$. But the eigenvalues of E are $3, 8, 10$ and the eigenvalues of F are $4, 5, 12$, so the matrices are not similar.

   G and H have the same sets of eigenvalues. But G is the identity and so is only similar to itself.

Diagonalization

For each of the following matrices, determine whether it is diagonal. If it is, diagonalize it, then computeA^5.

1. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

   Solution: A has eigenvalues $7, 3$ with eigenvectors $(1,1), (-1,1)$. This gives us

   $$\begin{aligned} U &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ U^{-1} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ D = U^{-1}AU &= \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \\ A^5 = UD^5U^{-1} &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}^5 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 16807 & 0 \\ 0 & 243 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 8525 & 8282 \\ 8282 & 8525 \end{bmatrix}. \end{aligned}$$

2. $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$

   Solution: The eigenvalues are $2, -1$ with corresponding eigenvectors $(1,1), (2,1)$. We have

$$\begin{aligned} U &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \\ U^{-1} &= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \\ D = U^{-1}AU &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \\ A^5 = UD^5U^{-1} &= U \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} U^{-1} = \begin{bmatrix} -34 & 66 \\ -33 & 65 \end{bmatrix}. \end{aligned}$$

3. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

   Solution: The only eigenvalue is 3, and the corresponding eigenvector is $(1,0,0)$. Thus the eigenvectors do not span $\mathbb{R}^3$ and so the matrix is not diagonalizable.

4. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

   Solution: The eigenvalues are $2, -1, 1$ with corresponding eigenvectors $(1,1,1), (-1,-1,2), (-1,1,0)$. We compute

$$\begin{aligned} U &= \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \\ U^{-1} &= \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & 2 \\ -3 & 3 & 0 \end{bmatrix} \\ D = U^{-1}AU &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ A^5 = UD^5U^{-1} &= U \begin{bmatrix} 32 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{-1} = \begin{bmatrix} 11 & 10 & 11\;10 & 11 & 11 \\ 11 & 11 & 10 & & \end{bmatrix}. \end{aligned}$$

5. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

   Solution: The eigenvalues are $2, 1, 1$ with corresponding eigenvectors $(0,1,0)$ and $(0,-1,1)$. The eigenvectors don't span, so the matrix is not diagonalizable.

6. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

   Solution:

A has eigenvalues $3, 1, 1$ with corresponding eigenvectors $(1,1,1), (-1,0,1), (0,1,0)$. Then we have

$$U = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$U^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$
$$D = U^{-1}AU = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A^5 = UD^5U^{-1} = U\begin{bmatrix} 243 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}U^{-1} = \begin{bmatrix} 122 & 0 & 121 \\ 121 & 1 & 121 \\ 121 & 0 & 122 \end{bmatrix}.$$

Orthogonality and Projection

1. Suppose $\|\mathbf{u}\| = 3, \|\mathbf{u}+\mathbf{v}\| = 4, \|\mathbf{u}-\mathbf{v}\| = 6$. Find $\|\mathbf{v}\|$.

   Solution: We have

$$\begin{aligned} 9 &= \langle \mathbf{u}, \mathbf{u} \rangle \\ 16 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ 36 &= \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle - 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ 52 &= 2\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{v}, \mathbf{v} \rangle = 2 \cdot 9 + 2\langle \mathbf{v}, \mathbf{v} \rangle \\ 34 &= 2\langle \mathbf{v}, \mathbf{v} \rangle \\ \sqrt{17} &= \|\mathbf{v}\|. \end{aligned}$$

2. Find the orthogonal complement (in $\mathbb{R}^n$) of the following spaces:

$$\begin{aligned} W &= \{(2t, -t) : t \in \mathbb{R}\} \\ W &= \text{span}\{(2,-1,3)\} \\ W &= \{(t,-t,3t) : t \in \mathbb{R}\} \\ W &= \text{span}\{(1,-1,3,-2), (0,1,-2,1)\}. \end{aligned}$$

   Solution:

$$\begin{aligned} W^\perp &= \text{span}\{(1,2)\} \\ W^\perp &= \text{span}\{(1,2,0), (3,0,-2)\} \\ W^\perp &= \text{span}\{(1,1,0), (-3,0,1) \end{aligned}$$
$$\begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$
$$W^\perp = \text{span}\{(-1,2,1,0), (1,-1,0,1)\}.$$

3. Find the orthogonal decomposition of

   (a) $(7,-4)$ with respect to $\text{span}\{(1,1)\}$

   Solution:

$$\begin{bmatrix} 7 \\ -4 \end{bmatrix}_U = \text{proj}_{(1,1)}\begin{bmatrix} 7 \\ -4 \end{bmatrix} = \frac{(7,-4)\cdot(1,1)}{(1,1)\cdot(1,1)}\begin{bmatrix} 7 \\ -4 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$
$$\begin{bmatrix} 7 \\ -4 \end{bmatrix}_{U^\perp} = \begin{bmatrix} 7 \\ -4 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 11/2 \\ -11/2 \end{bmatrix}.$$

(b) $(1,2,3)$ with respect to $\text{span}\{(2,-2,1),(-1,1,4)\}$
Solution: The basis we have is orthogonal, so we can just project onto it.

$$\text{proj}_{(2,-2,1)}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{(1,2,3)\cdot(2,-2,1)}{(2,-2,1)\cdot(2,-2,1)}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \frac{1}{9}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}2/9\\-2/9\\1/9\end{bmatrix}$$

$$\text{proj}_{(-1,1,4)}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{(1,2,3)\cdot(-1,1,4)}{(-1,1,4)\cdot(-1,1,4)}\begin{bmatrix}-1\\1\\4\end{bmatrix} = \frac{13}{18}\begin{bmatrix}-1\\1\\4\end{bmatrix} = \begin{bmatrix}-13/18\\13/18\\26/9\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}_U = \begin{bmatrix}2/9\\-2/9\\1/9\end{bmatrix} + \begin{bmatrix}-13/18 & 13/18\\ \multicolumn{2}{c}{22/9}\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\3\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix}_{U^\perp} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}-1/2\\1/2\\3\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\0\end{bmatrix}.$$

(c) $(4,-2,3)$ with respect to $\text{span}\{(1,2,1),(1,-1,1)\}$
Solution:

$$\text{proj}_{(1,2,1)}\begin{bmatrix}4\\-2\\3\end{bmatrix} = \frac{3}{6}\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1/2\\1\\1/2\end{bmatrix}$$

$$\text{proj}_{(1,-1,1)}\begin{bmatrix}4\\-2\\3\end{bmatrix} = \frac{9}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\-3\\3\end{bmatrix}$$

$$\begin{bmatrix}4\\-2\\3\end{bmatrix}_U = \begin{bmatrix}1/2\\1\\1/2\end{bmatrix} + \begin{bmatrix}3\\-3\\3\end{bmatrix} = \begin{bmatrix}7/2\\-2\\7/2\end{bmatrix}$$

$$\begin{bmatrix}4\\-2\\3\end{bmatrix}_{U^\perp} = \begin{bmatrix}4\\-2\\3\end{bmatrix} - \begin{bmatrix}7/2\\-2\\7/2\end{bmatrix} = \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}.$$

(d) $(3,2,-3,4)$ with respect to $\text{span}\{(2,1,0,1),(0,-1,1,1)\}$.
Solution:

$$\text{proj}_{(2,1,0,1)}\begin{bmatrix}3\\2\\-3\\4\end{bmatrix} = \frac{12}{6}\begin{bmatrix}2\\1\\0\\1\end{bmatrix} = \begin{bmatrix}4\\2\\0\\2\end{bmatrix}$$

$$\text{proj}_{(0,-1,1,1)}\begin{bmatrix}3\\2\\-3\\4\end{bmatrix} = \frac{-1}{3}\begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1/3\\-1/3\\-1/3\end{bmatrix}$$

$$\begin{bmatrix}3\\2\\-3\\4\end{bmatrix}_U = \begin{bmatrix}4\\2\\0\\2\end{bmatrix} + \begin{bmatrix}0\\1/3\\-1/3\\-1/3\end{bmatrix} = \begin{bmatrix}4\\7/3\\-1/3\\5/3\end{bmatrix}$$

$$\begin{bmatrix}3\\2\\-3\\4\end{bmatrix}_{U^\perp} = \begin{bmatrix}3\\2\\-3\\4\end{bmatrix} - \begin{bmatrix}4\\7/3\\-1/3\\5/3\end{bmatrix} = \begin{bmatrix}-1\\-1/3\\-8/3\\7/3\end{bmatrix}.$$

(e) $(2,-1,5,6)$ with respect to $U = \text{span}\{(1,1,1,0),(1,0,-1,1)\}$.
Solution: We see that $(1,1,1,0)\cdot(1,0,-1,1) = 1-1 = 0$, so this is an orthonormal basis for U. We compute

$$\text{proj}_{(1,1,1,0)} \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} = \frac{(2,-1,5,6)\cdot(1,1,1,0)}{(1,1,1,0)\cdot(1,1,1,0)} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{proj}_{(1,0,-1,1)} \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} = \frac{(2,-1,5,6)\cdot(1,0,-1,1)}{(1,0,-1,1)\cdot(1,0,-1,1)} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix}_U = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix}_{U^\perp} = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 4 \\ 5 \end{bmatrix}.$$

We can check that the second vector is in fact in $U^\perp$ by taking the inner product with the two basis vectors for U.

4. Let $V = \mathcal{P}_2(x)$ and define $\langle f, g\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$.

(a) Find the projection of $3x - 4x^2$ onto the vector $1 + x + x^2$.
Solution:

$$\begin{aligned} \text{proj}_{1+x+x^2} 3x - 4x^2 &= \frac{\langle 3x-4x^2, 1+x+x^2\rangle}{\langle 1+x+x^2, 1+x+x^2\rangle}(1+x+x^2) \\ &= \frac{(-7)(1)+(0)(1)+(-1)(3)}{(1)(1)+(1)(1)+(3)(3)}(1+x+x^2) \\ &= \frac{10}{11}(1+x+x^2). \end{aligned}$$

(b) Find the orthogonal decomposition of $2+x$ with respect to the spaces $W = \text{span}\{5+x\}$ and $W^\perp = \text{span}\{2-3x^2, -2+5x+2x^2\}$. (You can assume that the space I gave you is in fact $W^\perp$. But you can also check yourself, for practice.)
Solution: We have to project $2+x$ onto either W or $W^\perp$. It'll be a lot simpler to project onto W since it's lower dimension and we already have an orthogonal basis, so that's what we do.

$$\begin{aligned} (2+x)_W &= \text{proj}_{5+x} 2+x \\ &= \frac{\langle 2+x, 5+x\rangle}{\langle 5+x, 5+x\rangle}(5+x) \\ &= \frac{(1)(4)+(2)(5)+(3)(6)}{(4)(4)+(5)(5)+(6)(6)}(5+x) \\ &= \frac{32}{77}(5+x). \end{aligned}$$

Then the projection into $W^\perp$ is

$$(2+x)_{W^\perp} = 2 + x - \frac{32}{77}(5+x) = \frac{-6}{77} + \frac{45}{77}x.$$

(c) Find the orthogonal decomposition of $3-3x+x^2$ with respect to $W=\{3-5x, 4x-3x^2\}$ and $W^\perp=\{2+3x+2x^2\}$.

Solution: In this case we almost certainly want to project onto $W^\perp$. We have

$$\begin{aligned}(3-3x+x^2)_{W^\perp} &= \operatorname{proj}_{2+3x+2x^2} 3-3x+x^2 \\ &= \frac{\langle 3-3x+x^2, 2+3x+2x^2\rangle}{\langle 2+3x+2x^2, 2+3x+2x^2\rangle}(2+3x+2x^2) \\ &= \frac{(7)(1)+(3)(2)+(1)(7)}{(1)(1)+(2)(2)+(7)(7)}(2+3x+2x^2) \\ &= \frac{20}{54}(2+3x+2x^2) = \frac{10}{27}(2+3x+2x^2).\end{aligned}$$

Then

$$(3-3x+x^2)_W = 3-3x+x^2 - \frac{10}{27}(2+3x+2x^2) = \frac{61}{27} - \frac{37}{9}x + \frac{7}{27}x^2.$$

(d) Find the orthogonal complement of $W=\{\alpha_0+\alpha_2x^2 : \alpha_0,\alpha_2\in\mathbb{R}\}$.

Solution: We know our orthogonal complement should be one-dimensional. We want to find all polynomials $\beta_0+\beta_1x+\beta_2x^2$ that are orthogonal to every polynomial in W, which just means we need to solve

$$\begin{aligned}&(\alpha_0+\alpha_2(-1)^2)(\beta_0+\beta_1(-1)+\beta_2(-1)^2) \\ &\quad+(\alpha_0+\alpha_2(0)^2)(\beta_0+\beta_1(0)+\beta_2(0)^2) \\ &\quad+(\alpha_0+\alpha_2(1)^2)(\beta_0+\beta_1(1)+\beta_2(1)^2) \\ &\qquad=0.\end{aligned}$$

This simplifes to

$$\begin{aligned}0 &= (\alpha_0+\alpha_2)(\beta_0-\beta_1+\beta_2)+\alpha_0\beta_0+(\alpha_0+\alpha_2)(\beta_0+\beta_1+\beta_2) \\ &= 2(\alpha_0+\alpha_2)(\beta_0+\beta_2)+\alpha_0\beta_0 \\ &= 3\alpha_0\beta_0+2\alpha_0\beta_2+2\alpha_2\beta_0+2\alpha_2\beta_2.\end{aligned}$$

At this point we can do one of two things.

First, we could just solve these equations. This needs to hold for *any* α_0,α_2. So if we set $\alpha_0=0,\alpha_2=1$, we get $2\beta_0+2\beta_2=0$; and if we set $\alpha_0=1,\alpha_2=0$, we get $3\beta_0+2\beta_2=0$. Together, this implies that $\beta_0=\beta_2=0$. Thus $W^\perp=\{\beta_1x:\beta_1\in\mathbb{R}\}$.

Second, we could notice that we eliminated β_1 from our equations entirely, so β_1 *must* be a free parameter, and β_0 and β_2 can't depend on it. Since we know our space is one-dimensional, that's the only free parameter, and so there must be some fixed constant β_0,β_2 that work. It's easy to check that $x\in W^\perp$, so we can see that $W^\perp=\{\beta_1x:\beta_1\in\mathbb{R}\}$.