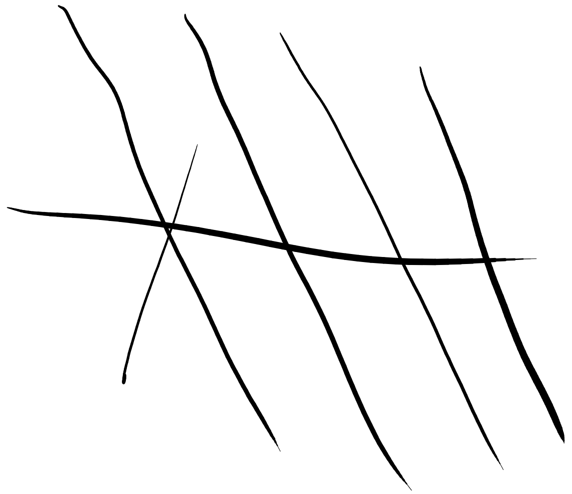
$\mathbb{Q}^n$

$\{x + \mathbb{Q}^n \mid x \in \mathbb{R}^n\}$

I can pick one $x \in \mathbb{R}^n$ for each translate

define $E = \{x\} = \mathbb{R}^n / \mathbb{Q}^n$

$(x_1, y_1) \sim (x_2, y_2)$ if $(x_1 - y_1) = (x_2 - y_2)$



$$\mathbb{R}^n = \bigcup_{x \in E} x + \mathbb{Q}^n \quad \text{disjoint}$$

$$\boxed{= \bigcup_{z \in \mathbb{Q}^n} z + E} \quad \text{disjoint}$$

$\mathbb{R}^n = \bigcup_{z \in \mathbb{Q}^n} z+E$ disjoint, countable

$$\lambda^*(\mathbb{R}^n) = \lambda^*\left(\bigcup z+E\right) \leq \sum_{z \in \mathbb{Q}^n} \lambda^*(z+E) = \sum_{z \in \mathbb{Q}^n} \lambda^*(E)$$

$\lambda^*(E) > 0$

Let $K \subseteq E$ fix $D = B_1(0) \cap \mathbb{Q}^n$

$$\bigcup_{r \in D} r+K \subseteq \bigcup_{r \in D} r+E$$

$$\lambda\Big(\bigcup_{r\in D} r+K\Big)=\sum_{r\in D}\lambda(r+K)=\sum_{r\in D}\lambda(K)$$

A

∞

since D, K bdd

Thus $\lambda(K)=0$.

if $\lambda(K)>0$

then $\sum \lambda(K)=\infty$.

Thus $\lambda(K)=0$.

So $\lambda_*(E)=0$.

K bdd $\quad K\subseteq B_r(0)$

$K+D\subseteq B_{r+1}(0)$

$\lambda(K+D)\le\lambda(B_{r+1}(0))<\infty$

If $A \subseteq \mathbb{R}^n$, $\lambda(A) > 0$

then $\exists\, B \subseteq A$ s.t. $B \notin \mathcal{L}$.

$$A = \bigcup_{z \in \mathbb{Q}^n} \left(\underline{(z+E) \cap A} \right) = \mathbb{R}^n \cap A$$

$$\bigcup (A_i \cap B) = \left(\bigcup A_i \right) \cap B$$

at least one $x_0 \in \mathbb{Q}^n$ s.t.

$$\lambda^*((x_0 + E) \cap A) > 0.$$

$(x_0 + E) \cap A = B$

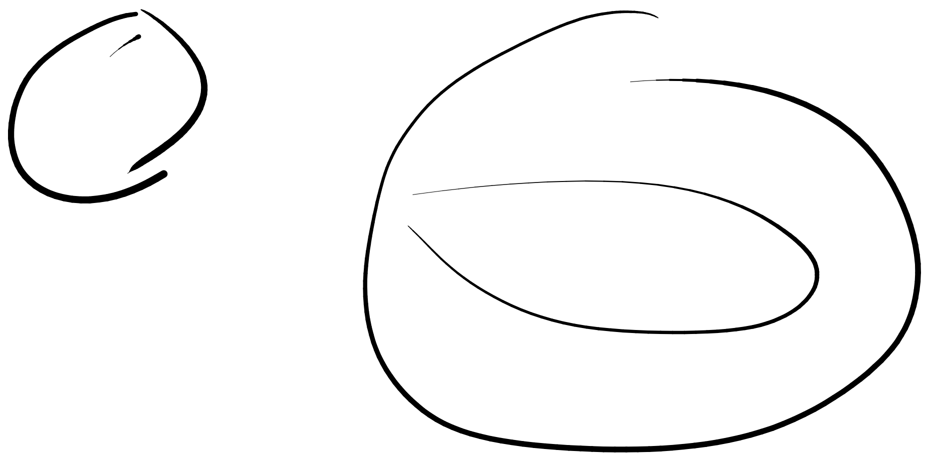
but $\lambda_*(B) = 0$

Banach-Tarski

Let $A, B \subseteq \mathbb{R}^3$ bdd

non-empty interior

A B



con Write $A = \bigcup_{k=1}^{n} A_k$, $B = \bigcup_{k=1}^{n} B_k$ disjoint

define Φ_k s.t.

$$\Phi_k(A_k) = B_k.$$