$$L^p(X, \mathcal{M}, \mu) = \{ f \mid \int_X |f|^p d\mu < \infty \}$$

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}.$$

Dfn: if $1 < p < \infty$, the Hölder conjugate of p is $q = \frac{p}{p-1}$, s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Hölder's inequality: $f \in L^p$, $g \in L^q$, then $fg \in L^1$

and $\left| \int_X fg \, d\mu \right| \leq \|f\|_p \|g\|_q$.

$p = 1$, $q = \infty$

$p = 2$, $q = 2$

$p = 3$, $q = 2/3$

Minkowski's inequality: $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

as $p \to \infty$, largest outputs are the only ones that matter.

Dfn: f meas. f is <u>essentially bounded</u> if

$\exists M < \infty$ s.t. $|f(x)| < M$ for almost all x.

$$f(x) = \begin{cases} x & x \in N \\ 1 & \text{otherwise.} \end{cases} \quad \text{"="} \quad f(x) = 1.$$

$L^\infty(X, \mathcal{M}, \mu)$ is the space of essentially bdd fns.

define $\|f\|_\infty = \inf\{M \mid |f(x)| < M \text{ a.e.}\}$

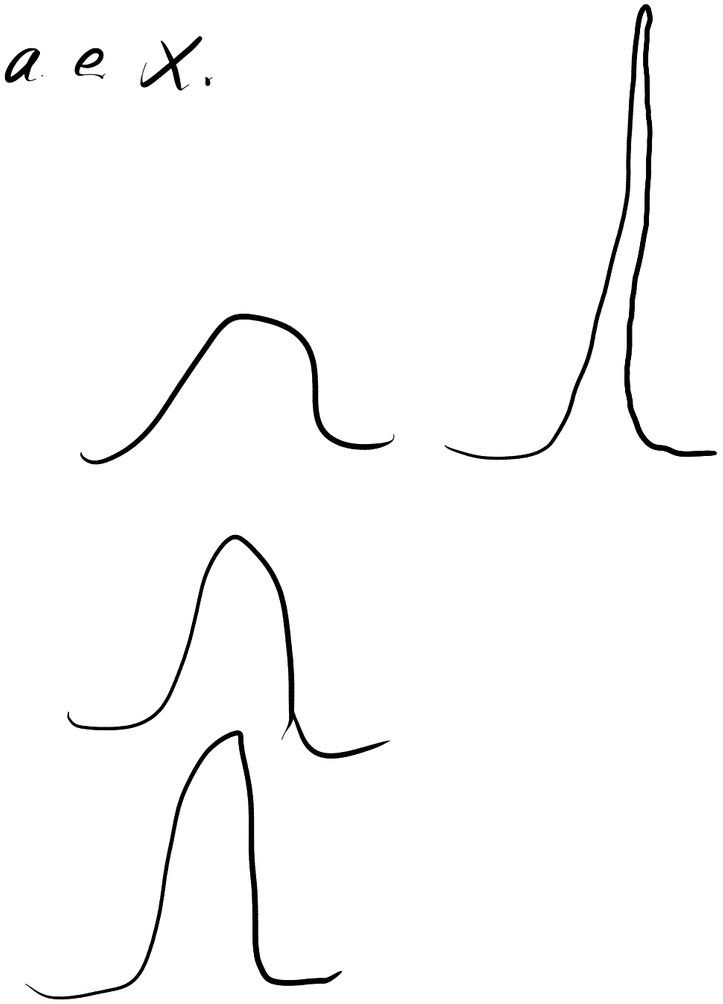
$\mathbb{R}^n = \{f : \{1, \dots, n\} \to \mathbb{R}\}$, $\int_n f \, d\mu = f(1)\cdot 1 + f(2)\cdot 1 + \dots + f(n)\cdot 1.$

$$\|f\|_2 = \sqrt{\int_n |f|^2 d\mu} = \sqrt{|f(1)|^2 + |f(2)|^2 + \dots + |f(n)|^2}$$

Prop: if $f \in L^\infty$, $|f(x)| \le \|f\|_\infty$ for a.e. x.

Prop: if $f \in L^r$ for some $r < \infty$, then

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty .$$



Thm (Riesz-Fischer): for $1 \le p < \infty$, $L^p(X)$ is a Banach space.
<u>Complete</u>

Pf/ Let f_n be Cauchy.

We can find a subsequence s.t. $\|f_{k+1} - f_k\|_p \le 2^{-k}$. WTS this converges

$$F = |f_1| + \sum_{j=1}^{\infty} |f_{j+1} - f_j|$$

$$\|F\|_p \le \|f_1\|_p + \sum_{j=1}^{\infty} \|f_{j+1} - f_j\|_p \le \|f_1\|_p + 1 < \infty.$$

So $\int |F| d\mu < \infty$, so F is finite a.e.

So $|f_1| + \sum |f_{j+1} - f_j| < \infty$ a.e, so $f_1 + \sum_{j=1}^{\infty} (f_{j+1} - f_j)$ converges absolutely.

define $f(x) = f_1 + \sum_{j=1}^{\infty} f_{j+1} - f_j$ when it converges.

$$f(x) = f_1 + \sum_{j=1}^{\infty} f_{j+1} - f_j$$

$$\|f - f_K\|_p = \left\| \sum_{j=K}^{\infty} f_{j+1} - f_j \right\|_p \le \sum_{j=K}^{\infty} \|f_{j+1} - f_j\|_p \le 2^{1-K} \to 0.$$

thus $f_K \to f$ in $\|\cdot\|_p$ norm.

Cor: If $f_K \to f$ in L^p

then $\exists$ subsequence that $\to f$ ptwise a.e.

- $C_B(\mathbb{R}^n)$ cts bounded fns $f: \mathbb{R}^n \to \mathbb{R}$

  $\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|$ Banach space $\subseteq L^\infty$

- $C_0(\mathbb{R}^n)$: cts $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $\lim_{|x| \to \infty} f(x) = 0$.

  $\subseteq C_B(\mathbb{R})$ sup norm.

  closed subset of complete space, so complete

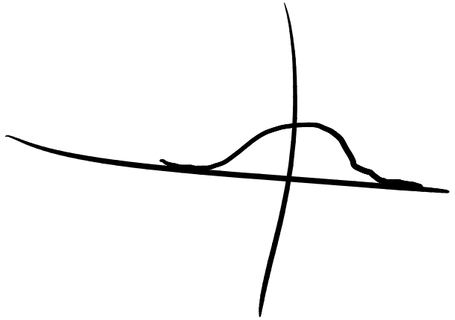
- $C_c(\mathbb{R}^n)$ cts fns w/ compact support

  i.e. $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $\exists K_f$ cpct and $f(x) = 0$ $\forall x \in K^c$.

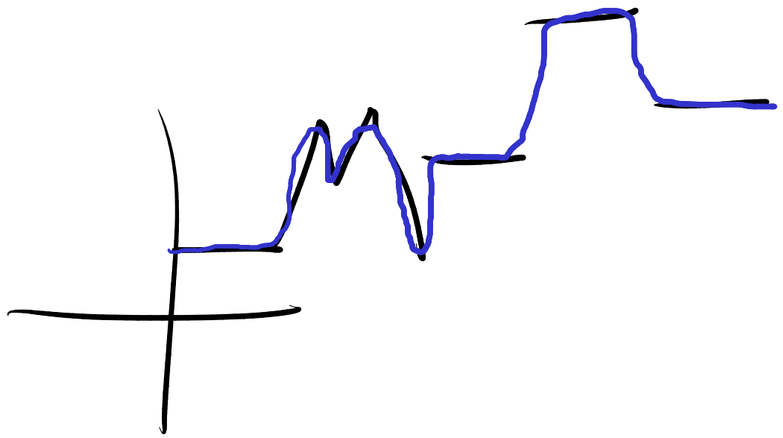
  $\subseteq C_0(\mathbb{R}^n)$, so use sup norm.

  $C_c(\mathbb{R}^n)$ dense in $C_0(\mathbb{R}^n)$

Def: $C_c^\infty(\mathbb{R}^n)$ $f: \mathbb{R}^n \to \mathbb{R}$ smooth w/ cpct support.



Thm: Assume $1 \le p < \infty$

Then $C_c^\infty(\mathbb{R}^n)$ are dense in $L^p(\mathbb{R}^n)$.

$e^{-\frac{1}{x^2}}$