$L^p(X) = \{ f \mid \int |f|^p \, d\mu < \infty \}$

Prop: If $0 < \mu(X) < \infty$, $1 \le p \le q < \infty$. Then $L^q \subseteq L^p$

Pf/ $\int_X |f|^p \, d\mu \le \int_X |f|^q + 1 \, d\mu = \int_X |f|^q \, d\mu + \mu(X) < \infty$

Ex: $\ell^p(R) = \{ (x_1, \ldots) \mid x_i \in \mathbb{R} \}$

$$\|(x_n)\|_p = \left(\sum_{j=1}^{\infty} |x_n|^p \right)^{1/p}$$

$$\ell^p(\mathbb{R}) = L^p(\mathbb{N}, 2^{\mathbb{N}}, \text{counting measure})$$

for $L^p(\mathbb{R}^n)$ can find <u>separable basis</u> $\{g_1, \ldots\}$

s.t. $f = \sum_{j=1}^{\infty} \alpha_j g_j$

$L^p \to \ell^p$

$f \mapsto (\alpha_1, \ldots)$

Functionals and innerproducts

V a VS, $f: V \to \mathbb{R}$, f is a linear <u>functional</u>.

The <u>dual</u> of V is the VS of cts linear functionals. V^*

$(X, \mathcal{M}, \mu)$ $f \in L^1(X)$. $f \mapsto \int_X f \, d\mu$

every μ gives a new LF <u>on a new space</u>

Fix $(X, \mathcal{M}, \mu)$ new measure $A \mapsto \int_A g \, d\mu$

Fix $g : X \to \mathbb{R}$, "can" compute $\int_X fg \, d\mu$.

If $fg \in L^1(X)$, then $\int_X fg \, d\mu \in \mathbb{R}$

Hölder: If $\frac{1}{p} + \frac{1}{q} = 1$, then $\left| \int_X fg \, d\mu \right| \leq \|f\|_p \|g\|_q$

If $g \in L^q$ we have a functional on L^p

$$f \mapsto \int_X fg \, d\mu$$

Inclusion $L^q \hookrightarrow L^p(X)^*$

$$g \mapsto \left(f \mapsto \int_X fg \, d\mu\right)$$

But actually $L^q \xrightarrow{\sim} L^p(X)^*$

i.e. every cts linear $L^p(X) \to \mathbb{R}$ comes from a $g \in L^q$.

$$T : L^p \times L^q \to \mathbb{R}$$

$$(f, g) \mapsto \int_X fg \, d\mu$$

If $p = 1$, works if X is σ-finite

If $p = \infty$, the dual of L^∞ is $>> L^1$ given AoC

Best case: $p=q=2$.

Then $L^2(X) \cong L^2(X)^*$

$\langle , \rangle : L^2(X) \times L^2(X) \to \mathbb{R}$

Dfn: $f,g \in L^2$, the inner product of f and g is

$\langle f,g \rangle = \int_X fg \, d\mu$. Makes L^2 into Hilbert space.

- Bilinear: $f \mapsto \langle f,g \rangle$ and $g \mapsto \langle f,g \rangle$ are linear
- Symmetric: $\langle f,g \rangle = \langle g,f \rangle$
- Pos Def: $\langle f,f \rangle \geq 0$, $\langle f,f \rangle = 0$ iff $f=0$ a.e.

IP induces a norm $\|f\|_{\langle,\rangle} = \sqrt{\langle f,f \rangle} = \sqrt{\int_X f^2 d\mu} = \|f\|_2$

$$\cos\theta = \frac{\langle f, g \rangle}{\|f\|_2 \, \|g\|_2}$$

Can prove If $\|f\|_p$ comes from any IP, then $p=2$.

IP => parallelogram rule:

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

f, g are orthogonal if $\langle f, g \rangle = 0$.

Ex: $X = [-\pi, \pi]$, λ.

$\{1, \sin(nx), \cos(nx) \mid n \in \mathbb{N}\}$ is orthogonal.

Prop: H a Hilbert space, $K \subseteq H$ complete. (Orthogonal decomposition)

for $h \in H$ can find a unique $k \in K$ s.t.

$\langle h-k, f \rangle = 0 \quad \forall f \in K$.

equivalently: $\|h-k\| = \inf\{\|h-f\| : f \in K\}$

Sketch: Set $\delta_K = \inf$.

Find $k \in K$ s.t. $\|h-k\| = \delta$ by choosing k_n s.t.

$\|h-k_n\| \to \delta_K$. (k_n) Cauchy by $\square$ law, so converges.

Written $H = K \oplus K^\perp$. (If $K = \mathrm{row}(A)$, $K^\perp = N(A)$).

Dfn: the projection of f onto g is

$$p = \mathrm{proj}_g f = \frac{\langle f, g \rangle}{\langle g, g \rangle} g$$

Then $f - p \perp g$.

Dfn: $\{f_1, \ldots\}$ is a <u>separable basis</u> for K, if every $g \in K$ can be uniquely written

$$g = \sum_{i=1}^{\infty} \alpha_i f_i \qquad \alpha_i \in \mathbb{R}.$$

orthogonal basis if $\langle f_i, f_j \rangle = 0$ for $i \neq j$.

Prop: set $p = \sum_{i=1}^{\infty} \frac{\langle g, f_i \rangle}{\langle f_i, f_i \rangle} f_i$

then $g - p \in K^{\perp}$.

Ex: $\{1, \sin(nx), \cos(nx)\}$ is an orthogonal basis for $L^2([-\pi, \pi])$.

$$\frac{1}{\pi} \int_{-\pi}^{\pi} g \sin(nx) \, dx$$