1 Euclidean Space

In this section we will review the basic properties of real Euclidean space. Most of them should be familiar to you from Analysis I (Math 310), but I'll collect them here so you can remember the important bits, and also have a useful reference.

1.1 Set Theory

We write $\mathbb{R}$ for the set of real numbers, and $\mathbb{R}^n$ for the set $\{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$ of n-tuples of real numbers. This is a special case of the *Cartesian product* of sets. If $A_1, \ldots, A_n$ are all sets, then

$$\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i\}.$$

We recall the set operations including union $A \cup B$, intersection $A \cap B$, set complement A^C, and set difference $A \setminus B$. We also have inclusion $A \subset B$ and containment $A \supset B$. If $A \cap B = \varnothing$ the empty set, then A and B are *disjoint*.

In this course we will often want to talk about unions and intersections of many sets. We often use I to stand for an *index set*, such that for each $i \in I$ we have a corresponding set A_i. Then we can write

$$\bigcup_{i \in I} A_i = \{a : \exists i \in I \text{ such that } a \in A_i\};$$
$$\bigcap_{i \in I} A_i = \{a : \forall i \in I, a \in A_i\}.$$

If the index set I is the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, instead we often write $\bigcup_{i=1}^{\infty} A_i$ or $\bigcap_{i=1}^{\infty} A_i$.

There are a couple of important principles about set intersection and union.

Fact 1.1 (De Morgan's Laws)**.** *Let I be an index set. Then*

$$\left(\bigcup_{i \in I} A_i\right)^C = \bigcap_{i \in I} A_i^C$$
$$\left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C.$$

Definition 1.2. Let $A_1, A_2, \ldots$ be a sequence of sets. We define

$$\limsup_{k\to\infty} A_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} A_k\right),$$
$$\liminf_{k\to\infty} A_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} A_k\right).$$

Proposition 1.3. *Let $A_1, A_2, \ldots,$ be a sequence of sets. Then*

$$\limsup_{k\to\infty} A_k = \{a : a \in A_k \textit{ for infinitely many } k \in \mathbb{N}\}.$$

Proof. If $a \in A_k$ for infinitely many $k \in \mathbb{N}$, then $a \in \bigcup_{k=j}^{\infty} A_k$ for any $j \in \mathbb{N}$. Thus $a \in \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k = \limsup_{k\to\infty} A_k$.

Conversely, if $a \in A_k$ for only finitely many k, then we can choose some j_0 larger than all of those k and then $a \notin \bigcup_{k=j_0}^{\infty} A_k$. Thus $a \notin \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k = \limsup_{k\to\infty} A_k$.

□

Exercise 1.4. *State and prove an analogue of Proposition 1.3 for* $\liminf_{k\to\infty} A_k$.

Now recall that we say a set A is *countable* if it is either finite or in bijection with the natural numbers. Informally, A is countable if you can list all of its elements in order. Recall that $\mathbb{N}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is not.

In fact, if $A_1, \ldots, A_n$ are all countable, then $\prod_{i=1}^{n} A_i$ is countable. And if I is a countable index set and A_i is countable for each $i \in I$, then $\bigcup_{i\in I} A_i$ is countable.

1.2 Topology and Metric in Euclidean Space

In order to understand sequences and sets, we need a sense of *topology*: we need to know which sets are "open", which tells us which points are close together. To really do analysis, we need something a bit stronger: we need a *metric*, which tells us how far apart two points are.

In Euclidean space we have something even better: a *norm*.

Definition 1.5. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We define the norm of x to be $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

The norm has the following important properties:

Fact 1.6. *Let $x, y \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Then*

- *(Positive definite)* $|x| \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0 = (0, \ldots, 0)$ *is the zero vector.*
- *(Scalars)* $|rx| = |r||x|$.
- *(Triangle Inequality)* $|x + y| \leq |x| + |y|$.

This norm gives us a *metric*:

Definition 1.7. Let $x, y \in \mathbb{R}^n$. We define the *distance* between x and y to be $d(x, y) = |x - y|$.

Exercise 1.8. *The distance* $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is a metric. That is, if* $x, y, z \in \mathbb{R}^n$, *then*

- *(Positive definite)* $d(x, y) \geq 0$, *and* $d(x, y) = 0$ *if and only if* $x = y$.
- *(Symmetry)* $d(x, y) = d(y, x)$.
- *(Triangle Inequality)* $d(x, z) \leq d(x, y) + d(y, z)$.

Recall that we can use this metric to define a convergent sequence:

Definition 1.9. Let $x \in \mathbb{R}^n$, and let $x_1, x_2, \ldots$ be a sequence of points in $\mathbb{R}^n$. We say that $\lim_{n\to\infty} x_n = x$ if, for every $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that if $n > N$ then $d(x_n, x) < \varepsilon$.

From the metric, we can also define open sets.

Definition 1.10. Let $x \in \mathbb{R}^n$ and $0 < r < \infty$. We define the *open ball with radius* r *and center* x to be

$$B_r(x) = B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}.$$

We define the *closed ball with radius* r *and center* x to be

$$\overline{B}_r(x) = \{y \in \mathbb{R}^n : d(x, y) \leq r\}.$$

If $x \in A \subset \mathbb{R}^n$, we say that x is an *interior point* of A if there is some $0 < r < \infty$ such that $B_r(x) \subset A$. We define the *interior* of A, denoted A° or $\mathring{A}$, to be the set of all interior points of A.

We say that A is *open* if every $x \in A$ is an interior point of A. A is open if and only if $A = A^\circ$.

Fact 1.11. *This definition of open sets defines a* topology *on* $\mathbb{R}^n$. *That is:*

- $\varnothing$ *and* $\mathbb{R}^n$ *are open.*

- *The union of* ***any*** *collection of open sets is open.*
- *The intersection of any* ***finite*** *collection of open sets is open.*

Exercise 1.12. *Find a collection of open sets whose intersection is not open.*

Exercise 1.13. *Prove that any open ball is an open set.*

We say that set is *closed* if its complement is open. Then

Fact 1.14.
- $\varnothing$ *and* $\mathbb{R}^n$ *are closed.*
- *The union of any* ***finite*** *collection of closed sets is closed.*
- *The intersection of* ***any*** *collection of closed sets is closed.*

Remark 1.15. Despite what you might think, "closed" and "open" are not opposites. Some sets are neither open nor closed. (Can you think of one?) Some sets are both open and closed. Topologists call those sets "clopen", because mathematicians have terrible senses of humor.

Definition 1.16. Let $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$. We say that x is a *limit point* of A if for every $r > 0$, there is a point $y \neq x$ such that $y \in A \cap B_r(x)$. That is, any open ball around x contains a point in A that is *not* x.

Fact 1.17. *x is a limit point of A if and only if $B_r(x)$ contains infinitely many points of A for any $r > 0$.*

A set A is closed if and only if it contains all its limit points.

We define the *closure* of A to be the set

$$\overline{A} = \{x : x \in A \text{ or } x \text{ is a limit point of } A\}.$$

We conclude this section with a note on notational convention, quoted directly from Jones:

Convention: Hereafter we shall strive for consistency in denoting open sets with the letter G and closed sets with the letter F. Obviously, any two letters would do, but tradition is on the side of G and F. In German the noun *Gebiet* means region, and in French the adjective *fermé* means closed.

1.3 Compact and Bounded

The definition of compactness is one of the most subtle and important in all of topology.

Definition 1.18. Let $A \subset \mathbb{R}^n$. Suppose that, whenever A is contained in a union of open sets, it is also contained in the union of some finite collection of those sets. Then we say A is *compact.*

We can write this in symbolic notation, which will be clearer in some ways and less clear than others. Suppose that whenever $A \subset \bigcup_{i \in I} G_i$ and each G_i is open, then there exist $i_1, \ldots, i_N \in I$ such that $A \subset \bigcup_{k=1}^N G_{i_k}$. Then A is compact.

We can get a grasp of this definition by seeing how it applies to a few easy cases:

Exercise 1.19. *Prove that:*

- $\varnothing$ *is compact.*
- *Any finite set is compact.*
- *If A, B are compact, then so is $A \cup B$.*
- *$B(x, r)$ is not compact.*
- *$\mathbb{R}^n$ is not compact.*

However, this definition is often unwieldy. Fortunately, in the case of Euclidean space specifically, there is a much easier criterion to check.

Definition 1.20. Let $A \subset \mathbb{R}^n$. We say that A is *bounded* if there is some $x \in \mathbb{R}^n$ and some $r > 0$ so that $A \subset B_r(x)$.

Theorem 1.21 (Heine-Borel)**.** *Let $A \subset \mathbb{R}^n$. Then A is compact if and only if it is closed and bounded.*

Proof. Please don't make me prove this. It's kinda tedious. □

There is one more way to think about compactness: we can relate it to sequence convergence.

Theorem 1.22 (Bolzano-Weierstrass)**.** *Every bounded infinite subset of $\mathbb{R}^n$ has a limit point.*

Proof. Let A be a bounded subset of $\mathbb{R}^n$ with no limit points. We shall prove it is finite.

Since A has no limit points, it contains all its limit points, and thus is closed. Since it is closed and bounded, it is compact.

If $x \in A$, then since x is not a limit point of A, there is some r_x so that $B_{r_x}(x) \cap A = \{x\}$. Now clearly $A \subset \bigcup_{x\in A} B_{r_x}(x)$; and since A is compact, that means there is some finite set I so that $A \subset \bigcup_{x\in I} B_{r_x}(x) = \bigcup_{x\in I}\{x\}$ which is a finite set. Thus A is finite. □

Exercise 1.23. *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Definition 1.24. We say a set A is *sequentially compact* if every sequence in A has a convergent subsequence.

Fact 1.25. *Every compact set is sequentially compact.*

Every sequentially compact subset of $\mathbb{R}^n$ is compact.

Remark 1.26. In a metric space, compactness and sequential compactness are equivalent. Thus for our purposes they are interchangeable. But you can construct a sequentially compact topological space that is not compact.

We have one more family of results we wish to prove about compact sets. When we begin to talk about the volume or measure of a set, we will want to talk about compact subsets of open sets. So we will conclude by proving a couple results about how well we can fit compact sets inside open sets.

Lemma 1.27. *Let K be compact, and G_i be open such that $K \subset \bigcup_{i\in I} G_i$.*

Then there exists an $\varepsilon > 0$ such that: for every $x \in K$ there exists an $i \in I$ such that $B_\varepsilon(x) \subset G_i$.

(The number ε is known as the Lebesgue number *for the covering $\{G_i\}$.)*

Note very importantly that there's a uniformity condition here: we have one ε that works for every x, though each x may work for only one i.

Proof. For each $x \in K$, there is an i_x such that $x \in G_{i_x}$. Since G_{i_x} is open, we can pick an r_x so that $B_{2r_x}(x) \subset G_{i_x}$.

We know that $K \subset \bigcup_{x\in K} B_{r_x}(x)$, and since K is compact and open balls are open, we can pick a finite set $x_1, \ldots, x_N \in K$ so that

$$K \subset \bigcup_{j=1}^{N} B_{r_{x_j}}(x_j).$$

Take ε to be the minimum of these r_{x_j}.

Now suppose $x \in K$. Then there is a j such that $x \in B_{r_{x_j}}(x_j)$, and thus $d(x, x_j) < r_{x_j} < 2r_{x_j} - \varepsilon$. Thus $B_\varepsilon(x) \subset B_{2r_{x_j}}(x_j) \subset G_{i_{x_j}}$.

□

Corollary 1.28. *Let K be compact and G be open, with $K \subset G$. Then there is an $\varepsilon > 0$ such that for all $x \in K$, we have $B_\varepsilon(x) \subset G$.*

Corollary 1.29. *Let K be compact and F be closed, with $K \cap F = \varnothing$. Then there exists an $\varepsilon > 0$ such that for every $x \in K$ and $y \in F$ we have $d(x, y) \geq \varepsilon$.*

1.4 Functions and Continuity

Definition 1.30. Let $x_0 \in A \subset \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. We say that f is *continuous* at x_0 if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in A$ and $d(x, x_0) < \delta$ then $d(f(x), f(x_0)) < \varepsilon$.

If f is continuous at x_0 for every $x_0 \in A$ then we say f is *continuous on A*.

It's very important to note that we're only worreid about $x \in A$ for the purposes of this definition. Thus whether a given function f is continuous at a given point x_0 or not can depend on what we give as the domain of f.

Exercise 1.31. *For any fixed $x \in \mathbb{R}^n$, the function $f(y) = d(x, y)$ is continuous on $\mathbb{R}^n$.*

Theorem 1.32. *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Then f is continuous if and only if for every open set $G \subset \mathbb{R}^m$, ther is an open set $H \subset \mathbb{R}^n$ such that $f^{-1}(G) = H \cap A$.*

This is basically what it means for a set to be open "in A": thus f is continuous if and only if $f^{-1}(G) \cap A$ is open in A.

In particular, if $f : \mathbb{R}^n \to \mathbb{R}^m$ then f is continuous if and only if $f^{-1}(G) \subset \mathbb{R}^n$ is open for any open $G \subset \mathbb{R}^m$.

Exercise 1.33. *State and prove an equivalent of theorem 1.32 for closed sets.*

A useful set of facts for working with this property:

Fact 1.34. *Let $f : A \to B$, and suppose $X_i \subseteq A, Y_i \subseteq B$. Then we have*

- $f^{-1}\left(\bigcup_{i \in I} Y_i\right) = \bigcup_{i \in I} f^{-1}(Y_i)$
- $f^{-1}\left(\bigcap_{i \in I} Y_i\right) = \bigcap_{i \in I} f^{-1}(Y_i)$

- $f\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} f(X_i)$
- $f\left(\bigcap_{i\in I} X_i\right) \subseteq \bigcup_{i\in I}(X_i)$.

Exercise 1.35. *Find a function $f : A \to B$ and a family of sets $X_i \subseteq A$ such that $f\left(\bigcap_{i\in I} X_i\right) \neq \bigcup_{i\in I}(X_i)$.*

Exercise 1.36. *Let $f : A \to B$. Then*

- *$f(f^{-1}(Y)) = Y \cap f(A)$ for any $A \subseteq B$.*
- *$f^{-1}(f(X)) \supset X$ for any $X \subseteq A$.*

Proposition 1.37. *Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Let K be a compact subset of A. Then $f(K)$ is compact.*

However, this doesn't work the other way.

Exercise 1.38. *Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a compact function $K \subseteq \mathbb{R}$ such that $f^{-1}(K)$ is not compact.*

Corollary 1.39. *Let $K \subseteq \mathbb{R}^n$ be compact and $f : K \to \mathbb{R}$ be continuous. then f attains a maximum and a minimum value on K.*

In math we're often interested in functions which are bijections—which show that two sets are in at least some sense equivalent. To preserve topological equivalence we want those invertible functions to be topologically "nice" functions, which in this cae means continuous.

Definition 1.40. Suppose $A \subseteq \mathbb{R}^n$ and $B \subseteq R^m$, and $f : A \to B$ is a bijection. If f and f^{-1} are continuous functions, we say that f is a *homeomorphism* from A to B.

Exercise 1.41. *Suppose A, B are open and $f : A \to B$ is a homeomorphism. Prove that f gives a bijection between the open subsets of A and the open subsets of B.*

Finally, recall that sometimes we want our functions to be not just continuous, but *uniformly* continuous.

Definition 1.42. Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Then f is *uniformly continuous* on A if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in A$ with $d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon$.

Every uniformly continuous function is continuous, but the converse isn't true. However:

Fact 1.43. *If K is compact and $f : K \to \mathbb{R}^m$ is continouous, then f is uniformly continuous.*

1.5 Distance from a Set

We have a clear definition of the distance between two points, but we often want to know the distance between a point and a set. This turns out to be a bit subtler but not too bad.

Definition 1.44. Let $A \subseteq \mathbb{R}^n$ be nonempty, and let $x \in \mathbb{R}^n$. The *distance* from x to A is the number

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

(Since this is a set of real numbers bounded below by 0, it has an infimum.)

Proposition 1.45. *If $A \subseteq \mathbb{R}^n$ is nonempty and $x \in \mathbb{R}^n$, then there is a $x_0 \in \overline{A}$ such that $d(x, A) = d(x, x_0)$.*

Proof. □

Corollary 1.46. *If $A \subseteq \mathbb{R}^n$ is closed and nonempty, and $x \in \mathbb{R}^n$, then there is a $x_0 \in A$ such that $d(x, A) = d(x, x_0)$. That is, there is a closest point to x in A.*

Exercise 1.47. *Let $A \subseteq \mathbb{R}^n$. Then $x \in \overline{A}$ if and only if $d(x, A) = 0$.*

Proposition 1.48. *Assume $A \neq \varnothing$. Then $d(x, A)$ is a continuous function of x.*

Proof. For $x, x' \in \mathbb{R}^n$, for any $y \in A$ we have that

$$d(x, A) \leq d(x, y) \leq d(x, x') + d(x', y).$$

Thus $d(x, A) - d(x, x') \leq d(x', y)$ for any $y \in A$. Thus

$$d(x, A) - d(x, x') \leq d(x', A).d(x, A) - d(x', A) \qquad \leq d(x, x').$$

By symmetry, we also must have

$$\begin{aligned} d(x', A) - d(x, A) &\leq d(x, x') \\ |d(x, A) - d(x', A)| &\leq d(x, x'). \end{aligned}$$

□

Theorem 1.49 (Bump Functions)**.** *Assume F is closed, G open, and $F \subseteq G \subseteq \mathbb{R}^n$. Then there is a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\begin{aligned} 0 \leq f(x) &\leq 1 && \forall x \in \mathbb{R}^n \\ f(x) &= 1 && \forall x \in F \\ f(x) &= 0 && \forall x \in G^c. \end{aligned}$$

That is, for any closed subset of an open set, we can write a continuous function that is 1 on the closed subset and 0 outside of the open set. Such a function is called an *Urysohn function* after Pavel Urysohn.

Proof. This is easy if either F or G^c is empty. Otherwise, we can define

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}.$$

We know that f is continuous by proposition 1.48, since the denominator is never zero. □

Definition 1.50. Let $\varnothing \neq A \subseteq \mathbb{R}^n$. The *diameter* of A is

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

We know that $0 \leq \operatorname{diam}(A) \leq \infty$.

Exercise 1.51. *Let $\varnothing \neq A \subseteq \mathbb{R}^n$. Then*

- $\operatorname{diam}(A) = 0$ *if and only if A contains exactly one point.*
- $\operatorname{diam}(A) < \infty$ *if and only if A is bounded.*
- $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.
- $\operatorname{diam}(B_r(x)) = 2r$.