Math 310 Fall 2018
Real Analysis HW 1 Solutions
Due Wednesday, January 29

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Prove that $\liminf A_k \subset \limsup A_k$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. State and prove an analogue of Proposition 1.3 for $\liminf_{k\to\infty} A_k$.

2. Prove that $\limsup(A_k \cup B_k) = \limsup A_k \cup \limsup B_k$.

3. A sequence of sets is *decreasing* if $A_1 \supset A_2 \supset A_3 \supset \ldots$. Prove that if the sequence is decreasing, then
$$\limsup A_k = \liminf A_k = \bigcap_{k=1}^{\infty} A_k.$$

4. Give an example of a decreasing sequence of nonempty sets of real numbers whose intersection is empty.

5. Prove that The distance $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a metric. That is, if $x, y, z \in \mathbb{R}^n$, then
   - (Positive definite) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
   - (Symmetry) $d(x, y) = d(y, x)$.
   - (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

6. Prove that $|d(x, y) - d(x, z)| \leq d(y, z)$. (You can assume that d is the Euclidean metric, but this holds for any metric and you only need to use the definition-of-metric properties to prove it.)

7. Prove that $B(x, r) \subset B(x', r')$ if and only if $d(x, x') \leq r' - r$.

8. Find a collection of open sets whose intersection is not open.

9. Find a collection of closed sets whose union is not closed.

10. Prove that any open ball is an open set.

11. Prove that:
    - $\varnothing$ is compact.
    - Any finite set is compact.
    - If A, B are compact, then so is $A \cup B$.
    - $B(x, r)$ is not compact.
    - $\mathbb{R}^n$ is not compact.
12. Prove that very bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.