Math 395 Spring 2020
Real Analysis HW 2 Solutions
Friday, February 7

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Suppose $\varnothing \neq A \subseteq B \subseteq \mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Prove that $d(x, A) \geq d(x, B)$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Prove that, for any fixed $x \in \mathbb{R}^n$, the function $f(y) = d(x, y)$ is continuous on $\mathbb{R}^n$.
2. State and prove an equivalent of theorem 1.32 for closed sets.
3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $G \subset \mathbb{R}$ such that $f(G)$ is not open. This is a converse to 1.32.
4. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed set $F \subset \mathbb{R}$ such that $f(F)$ is not closed.
5. Let $f : A \to B$. Prove that:
   - $f(f^{-1}(Y)) = Y \cap f(A)$ for any $A \subseteq B$.
   - $f^{-1}(f(X)) \supset X$ for any $X \subseteq A$.
6. Find an example of a function $f : A \to B$ and a set X where $f^{-1}(f(x)) \neq X$.
7. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a compact function $K \subseteq \mathbb{R}$ such that $f^{-1}(K)$ is not compact.
8. Suppose A, B are open and $f : A \to B$ is a homeomorphism. Prove that f gives a bijection between the open subsets of A and the open subsets of B.
9. Prove that $x \in \overline{A}$ if and only if $d(x, A) = 0$.
10. Prove that $d(x, A) = d(x, \overline{A})$.
11. Let $\varnothing \neq A \subseteq \mathbb{R}^n$. Then

- $\operatorname{diam}(A) = 0$ if and only if A contains exactly one point.
- $\operatorname{diam}(A) < \infty$ if and only if A is bounded.
- $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.
- $\operatorname{diam}(B_r(x)) = 2r$.

12. Find a set such that $\operatorname{diam}(A) \neq \operatorname{diam}(A^\circ)$.