Math 395 Spring 2020
Real Analysis HW 3
Friday, February 14

You may *not* discuss the starred problem with classmates, though you should of course feel free to discuss it with me as much as you like. Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Prove that if $K_1 \subset K_2$ then $\lambda(K_1) \leq \lambda(K_2)$.

For the remainder of these problems, I encourage you to collaborate with your classmates, as well as to discuss them with me.

1. Let $I \subset \mathbb{R}^n$ be a special rectangle. Prove that the following conditions are equivalent:
   (a) $\lambda(I) = 0$
   (b) $I^\circ = \varnothing$
   (c) I is contained in an affine subspace of $\mathbb{R}^n$ having dimension smaller than n. (An affine subspace is a set $\{x_0 + x : x \in E\}$ where E is a subspace and x_0 is a fixed point.)
2. If G is a bounded open set, prove that $\lambda(G) < \infty$.
3. Prove that $\lambda(\mathbb{R}^n) = \infty$.
4. Let
   $$G = \{(x, y) : 1 < x, 0 < y < 1/x\} \subset \mathbb{R}^2.$$
   Prove that $\lambda(G) = \infty$.
5. Let
   $$G = \{(x, y) : 0 < x, 0 < y < e^{-x}\} \subset \mathbb{R}^2.$$
   Prove that $\lambda(G) = 1$.
6. Let G_i for $i \in I$ be a collection of disjoint open sets in $\mathbb{R}^n$. Prove that only countably many of these sets are nonempty. (Hint: if G_i is not empty, show it contains a point of $\mathbb{Q}^n$.)
7. Prove that every nonempty open subset of $\mathbb{R}$ can be written as a countable disjoint union of open intervals $G = \bigcup k(a_k, b_k)$, and this expression is unique.

   Then conclude that $\lambda(G) = \sum_k (b_k - a_k)$.

8. Prove that the Cantor set C is compact. Then prove that $\lambda(C) = 0$.

9. Prove that $\frac{1}{4} \in C$, but that $\frac{1}{4}$ is not an endpoint of any G_k.

10. Let $\varepsilon > 0$. Prove that there is an open set $G \subseteq \mathbb{R}$ such that $Q \subseteq G$ and $\lambda(G) < \epsilon$.

11. Use the previous problem to show that if A is a countable set, then $\lambda^*(A) = 0$.

12. Let A_k be a disjoint family of subsets of $\mathbb{R}^n$. Prove that

$$\lambda_* \left(\bigcup_{k=1}^{\infty} A_k \right) \geq \sum_{k=1}^{\infty} \lambda_*(A_k).$$