Math 395 Spring 2020
Real Analysis 2 HW 4
Due Friday, March 27

Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Prove that $P(A|B)$ is a probability measure on B.

1. Prove that if $A \subset \mathbb{R}^n$ is countable then A is measurable and $\lambda(A) = 0$. Conclude that $\lambda(\mathbb{Q}) = 0$. Conclude that $\lambda([0,1] \setminus \mathbb{Q}$ is measurable and has measure 1. Demonstrate that there is a set with empty interior and positive measure.

2. Suppose $A \cup B$ is measurable, and $\lambda(A \cup B) = \lambda^*(A) + \lambda^*(B) < \infty$. Prove that A and B are measurable.

3. Let $X = \mathbb{R}$ and define $\mathcal{M} \subset 2^{\mathbb{R}}$ by $A \in \mathcal{M}$ if and only if either $A = \varnothing$, or A is a finite union of intervals of the form $[a, b)$ or the form $(-\infty, b)$. Prove that $\mathcal{M}$ is an algebra, but not a σ-algebra.

4. Give an example of two σ-algebras whose union is not a σ-algebra.

5. Prove that the class of Borel sets is *also* the σ-algebra generated by the collection of special rectangles.

   Hint: let $\mathcal{M}$ be the σ-algebra generated by special rectangles. Prove that all open sets are in $\mathcal{M}$. Conclude that $\mathcal{B} \subseteq M$.

   You *cannot* assume that every Borel set can be constructed from open sets by countably many set operations.

6. Prove that, if $E \subseteq A \in \overline{\mathcal{M}}$ and $\overline{\mu}(A) = 0$, then $E \in \overline{\mathcal{M}}$ and $\overline{\mu}(E) = 0$.

7. Prove the following facts about abstract measures:

   (a) If $A, B \in \mathcal{M}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

   (b) If $A_1, A_2, \dots \in \mathcal{M}$, then

   $$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

(c) If $A_1 \subseteq A_2 \subseteq \ldots$ are in $\mathcal{M}$ then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

8. Prove that, if $E \subseteq A \in \overline{\mathcal{M}}$ and $\overline{\mu}(A) = 0$, then $E \in \overline{\mathcal{M}}$ and $\overline{\mu}(E) = 0$.

9. Prove that $f : U \to V$ is affine if and only if there is a linear function $L : U \to V$ and a vector $v \in V$ such that $f(x) = v + L(x)$ for every $x \in U$. Further, this choice of L and v is unique.

10. Prove that a matrix is orthogonal if and only if the associated linear transformation is orthogonal.

11. Prove that if L is orthogonal, then $|\det L| = 1$. Hint: use theorem 3.5 and use $A = B(0, 1)$.

12. Prove that the translates of $\mathbb{Q}^n$ partition $\mathbb{R}^n$. That is, if $x, y \in \mathbb{R}^n$, then either $x+\mathbb{Q}^n = y + \mathbb{Q}^n$ or $(x + \mathbb{Q}^n) \cap (y + \mathbb{Q}^n) = \varnothing$.