Math 395 Spring 2020
Real Analysis 2 HW 5
Due Friday, April 3

Linguistic precision is important for this problem. Submit this problem on a separate, detached sheet of paper.

$\star$ **Redo Problem:** Let $A \subset X$. Prove that the characteristic function χ_A is $\mathcal{M}$-measurable if and only if $A \in \mathcal{M}$.

1. Prove that $A \subset \mathbb{R}^n$ is measurable if and only if there is a $B \in F_\sigma$ and a $C \in G_\delta$ such that $B \subseteq A \subseteq C$ and $C \setminus B$ is null.

2. Prove that there are disjoint subsets $A, B \subseteq \mathbb{R}^n$ such that

$$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$$
$$\lambda_*(A \cup B) > \lambda_*(A) + \lambda_*(B).$$

3. Let $A, B, C \subseteq \mathbb{R}^n$ such that $A \subseteq C$ and $\lambda(B \cap C) = 0$. Then A, B are not necessarily disjoint but they are separated in a measure theoretic sense. Prove that $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$.

4. Let $E \subseteq \mathbb{R}^n$ and $f : E \to \mathbb{R}$ be uniformly continuous. Prove that there is a unique function $F : \overline{E} \to \mathbb{R}$ such that F is continuous and $F(x) = f(x)$ for all $x \in E$.

5. If f is the Lebesgue function associated to some Cantor set A, prove that $f(1-x)+f(x) = 1$ for any x.

6. Let $\mathcal{M} = \{\varnothing, X\}$ and $\mathcal{N} = 2^X$. Describe explicitly the sets of $\mathcal{M}$-measurable functions and of $\mathcal{N}$-measurable functions.

7. If $f : X \to \mathbb{R}$ is a $\mathcal{M}$-measurable function and $f(x) \neq 0$ for any $x \in X$, prove that $\frac{1}{f}$ is $\mathcal{M}$-measurable.

8. Prove that a simple function s is measurable if and only if each set A_k is measurable.

9. Let $f : X \to \mathbb{R}$ be a measurable function. Prove that f_+ and f_- are measurable.