Math 395 Spring 2020
Real Analysis 2 HW 7
Due Friday, April 17

1. If f is measurable and $f = 0$ almost everywhere, prove that $f \in L^1$ and $\int f \, d\mu = 0$.

2. Suppose $g(x) = h(x)$ almost everywhere. Prove that g is measurable if and only if h is measurable.

   (This allows us to take a function f that is defined almost everywhere, and say it is measurable if an extension to all of X is measurable.)

3. Let f be an integrable function on a measure space $X, \mathcal{M}, \mu$, and suppose that for any measurable set $E \in \mathcal{M}$ we have that $\int_E f \, d\mu = 0$. Prove that $f(x) = 0$ almost everywhere.

4. Suppose we have a sequence of non-negative functions $0 \leq f_k \in L^1$, and set $f = \lim_{k\to\infty} f_k$. Prove that $\int_E f \, d\lambda = \lim_{k\to\infty} \int_E f_k \, d\lambda$.

5. Let X be a set, $\mathcal{M} = 2^X$, choose some $x_0 \in X$. Define a the Dirac delta measure $\delta_{x_0}(E) = \chi_E(x_0)$. Prove that any function $f : X \to \overline{\mathbb{R}}$ is measurable. Prove that $f \in L^1$ if and only if $f(x_0) \in \mathbb{R}$, and in that case, that $\int f \, d\delta_{x_0} = f(x_0)$.

6. If $f : X \to [0, \infty]$, and $\sum_{x\in X} f(x) < \infty$, prove that $\{x | f(x) > 0\}$ is countable. (Hint: how big is the set $\{x : f(x) > 1/k\}$?)

7. Let X be any set, $\mathcal{M} = 2^X$, and μ be the counting measure. If $f : X \to \overline{\mathbb{R}}$, prove that $f \in L^1$ if and only if $\sum_{x\in X} |f(x)| < \infty$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\overline{\mu}$ be the completion of μ. If $f : X \to \overline{\mathbb{R}}$ is μ-measurable, we know it must also be $\overline{\mu}$-measurable. Prove that $\int f \, d\overline{\mu} = \int f \, d\mu$.

   (Conversely, if g is $\overline{\mathcal{M}}$-measurable, it need not be $\mathcal{M}$-measurable. But there is a $\mathcal{M}$-measurable function f such that $f(x) = g(x)$ almost everywhere, and then $\int g \, d\overline{\mu} = \int f \, d\mu$.)

9. Let $E \in \mathcal{M}$ and assume $\lambda(E) = 0$. Prove that every function defined on E is measurable, and that $\int_E f \, d\mu = 0$ for any f defined on E.