Math 1231 Midterm Solutions

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1. You will have 75 minutes for this test.
2. You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
3. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.
4. This test has eight questions, over six pages. **You should not answer all eight questions.**
   (a) The first three problems are three pages, representing topics M1, M2, and M3, and you should do all three of them. They are worth twenty points each.
   (b) The remaining five problems represent topics S1 through S5. You should select **up to three** of these questions and answer them. Your two best will be worth ten points each; the third will be worth up to five bonus points. It is better to answer two questions well than three questions poorly.
   (c) If you answer more than three out of the last five problems, we'll ignore the extra, so please pick and choose.
   (d) If you perform well on a question on this test it will update your mastery scores. Achieving a 18/20 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

1		5	
2		6	
3		7	
4		8	
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Problem 1 (M1)**.** Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 5} \frac{1}{x-5} - \frac{5}{x^2-5x}$$

Solution:

$$\lim_{x\to 5} \frac{1}{x-5} - \frac{5}{x^2-5x} = \lim_{x\to 5} \frac{x-5}{x^2-5x} = \lim_{x\to 5} \frac{1}{x} = \frac{1}{5}.$$

(b)

$$\lim_{x\to 1} \frac{\sqrt{x+1}-2}{x-1}$$

Solution:

The limit of the top is $\sqrt{2}-2$ and the limit of the bottom is zero, so the limit is $\pm\infty$. Since the bottom can be positive or negative, we can't be any more specific.

We could, if we wanted, multiply by the conjugate. That would give us

$$\lim_{x\to 1} \frac{\sqrt{x+1}-2}{x-1} = \lim_{x\to 1} \frac{x-3}{(x-1)(\sqrt{x+1}+2)}$$

and now the numerator goes to -2 and the bottom goes to zero. So again we see the limit is $\pm\infty$.

(c)

$$\lim_{x\to+\infty} \frac{\sqrt{3x^5+2x}}{x^{5/2}-x^{3/2}+1}$$

Solution:

$$\lim_{x\to+\infty} \frac{\sqrt{3x^5+2x}}{x^{5/2}-x^{3/2}+1} = \lim_{x\to+\infty} \frac{\sqrt{3+2/x^4}}{1-1/x+1/x^{5/2}} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

(d)

$$\lim_{x\to-2} \frac{\tan(2x+4)}{x+2}$$

Solution:

$$\lim_{x\to-2} \frac{\tan(2x+4)}{x+2} = \lim_{x\to-2} \frac{\sin(2x+4))\cdot 2}{(2x+4)\cdot\cos(2x+4)} = 1\cdot\frac{2}{1} = 2.$$

Problem 2 (M2)**.** (a) Compute the derivative of $f(x) = \tan^3(\sec^2(x^2+1))$, using methods we've developed in class.

Solution:

$$f'(x) = 3\tan^2(\sec^2(x^2+1))\sec^2(\sec^2(x^2+1))\cdot 2\sec(x^2+1)\sec(x^2+1)\tan(x^2+1)\cdot 2x$$

(b) If $x^3 = xy^2 + 3$, find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution:

$$\begin{aligned}
3x^2 &= y^2 + 2xyy' \\
3x^2 - y^2 &= 2xyy' \\
y' &= \frac{3x^2 - y^2}{2xy} \\
y' &= \frac{(6x - 2yy')2xy - 2(y + xy')(3x^2 - y^2)}{4x^2y^2} \\
&= \frac{\left(6x - 2y\frac{3x^2-y^2}{2xy}\right)2xy - 2\left(y + x\frac{3x^2-y^2}{2xy}\right)(3x^2 - y^2)}{4x^2y^2}.
\end{aligned}$$

Problem 3 (M3)**.**

(a) Find a tangent line to the curve given by $x^3y - x^2y^2 = -2$ at the point $(1, 2)$.

Solution: We use implicit differentiation, and find that

$$\begin{aligned}
3x^2y + x^3y' - 2xy^2 - 2x^2yy' &= 0 \\
3x^2y - 2xy^2 &= 2x^2yy' - x^3y' \\
y' &= \frac{3x^2y - 2xy^2}{2x^2y - x^3}
\end{aligned}$$

Thus at the point $(1, 2)$ we have

$$\frac{dy}{dx} = \frac{6 - 8}{4 - 1} = \frac{-2}{3}.$$

Thus the equation of our tangent line is

$$\begin{aligned}
y &= y_0 + m(x - x_0) \\
y &= 2 + \frac{-2}{3}(x - 1).
\end{aligned}$$

Alternatively, we could have substituted in much earlier. We can compute that at the point $(1, 2)$, weh ave

$$\begin{aligned}
3x^2y + x^3y' - 2xy^2 - 2x^2yy' &= 0 \\
6 + y' - 8 - 4y' &= 0 \\
-3y' &= 2 \\
y' &= -2/3.
\end{aligned}$$

As before, we get the line

$$y = 2 + \frac{-2}{3}(x - 1).$$

(b) Give a formula for the linear approximation of the function $f(x) = \sqrt{3x + 1}$ near the point $a = 5$, and use it to estimate $f(4.9)$.

Solution: We calculate that $f(5) = \sqrt{16} = 4$, and $f'(x) = \frac{1}{2}(3x + 1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x+1}}$ so $f'(5) = \frac{3}{8}$. Then we have

$$\begin{aligned}
f(x) &\approx 4 + \frac{3}{8}(x - 5) \\
f(4.9) &\approx 4 + \frac{3}{8}(4.9 - 5) = 4 - \frac{3}{80} = \frac{317}{80}.
\end{aligned}$$

(This is exactly 3.9625, compared to the true answer of roughly 3.96232.)

Problem 4 (S1)**.** Explicitly naming each limit law you use, compute

$\lim_{x\to 2} 3x + \dfrac{x^2-3x+2}{x-2} =$

Solution:

$$\begin{aligned}
\lim_{x\to 2} 3x + \frac{x^2-3x+2}{x-2} &= \lim_{x\to 2} 3x + \lim_{x\to 2} \frac{x^2-3x+2}{x-2} && \text{addition}\\
&= 3\lim_{x\to 2} x + \lim_{x\to 2} \frac{(x-1)(x-2)}{x-2} && \text{scalar products (and basic algebra)}\\
&= 3\lim_{x\to 2} x + \lim_{x\to 2} x-1 && \text{almost identical functions}\\
&= 3\lim_{x\to 2} x + \lim_{x\to 2} x - \lim_{x\to 2} 1 && \text{addition}\\
&= 3\cdot 2 + 2 - \lim_{x\to 2} 1 && \text{identity}\\
&= 6+2-1 && \text{constants}
\end{aligned}$$

The important trick is that you *cannot* use the quotient law on this problem; it doesn't work if you'd be dividing by zero. You have to use almost identical functions there.

You can't use AIF after the quotient rule, because the whole point is that only works in the limit. We know $\lim_{x\to 2} x-2 = 0$, so

$$\frac{\lim_{x\to 2} x-2}{\lim_{x\to 2} x-2}\text{"} = \text{"}\frac{0}{0}$$

is undefined. But

$$\lim_{x\to 2} \frac{x-2}{x-2} = \lim_{x\to 2} 1 = 1.$$

Problem 5 (S2)**.** Show that $\lim_{x\to 3} \dfrac{x-3}{3+\sin\left(\frac{1}{x-3}\right)} = 0$.

Solution: We know that

$$\begin{aligned}
-1 \leq \sin\left(\frac{1}{x-3}\right) &\leq 1\\
2 \leq 3+\sin\left(\frac{1}{x-3}\right) &\leq 4\\
\frac{1}{4} \leq \frac{1}{3+\sin\left(\frac{1}{x-3}\right)} &\leq \frac{1}{2}\\
\frac{|x-3|}{4} \leq \frac{|x-3|}{3+\sin\left(\frac{1}{x-3}\right)} &\leq \frac{|x-3|}{2}
\end{aligned}$$

Since $\lim_{x\to 3} |x-3|/4 = 0$ and $\lim_{x\to 3} |x-3|/2 = 0$, by the Squeeze Theorem, we have $\lim_{x\to 3} \frac{x-3}{3+\sin\left(\frac{1}{x-3}\right)} = 0$.

Problem 6 (S3)**. Directly from the definition of derivative,** compute the derivative of $f(x) = \frac{3}{x+2}$ at $a = 1$.

Solution:

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{3+h}-1}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{3+h}-\frac{3+h}{3+h}}{h} \\ &= \lim_{h\to 0} \frac{-h}{h(3+h)} \\ &= \lim_{h\to 0} \frac{-1}{3+h} = \frac{-1}{3}. \end{aligned}$$

Problem 7 (S4)**.** Suppose that $p(t) = 10-2t$ is momentum (in kilogram meters per second) of a ball thrown directly upwards, as a function of time (in seconds).

(i) What does the derivative $p'(t)$ represent, and what are its units?

Solution: The derivative is the rate at which the momentum decreases over time. Its units are kilogram meters per second per second, or kilogram meters per second squared.

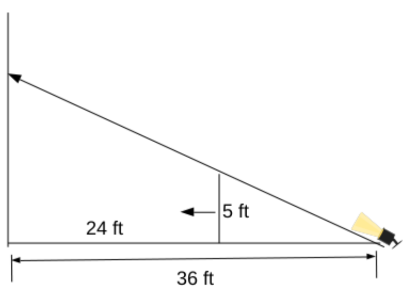
(Bonus fact: note that this is also the units of mass times acceleration. If that seems like force to you, you're right! The original formulation of Newton's second law was $F = \frac{dp}{dt}$. Which means that, just like the derivative of velocity is acceleration, the derivative of momentum is force.)

(ii) Calculate $p'(3)$. What does this tell you?

Solution: $p'(t) = -2$ so $p'(3) = -2$. This means that the momentum decreases by about 2 kilogram meters per second every second. In particular, between time $t = 3$ and time $t = 4$ the momentum will decrease by about 2 kilogram-meters per second.

(The derivative is constant, but that does *not* mean that the momentum doesn't change. It means the rate at which the momentum is changing doesn't change, but that is importantly different.)

Problem 8 (S5)**.** A spot light is on the ground 36 ft away from a wall and a 5 ft tall person is walking towards the wall at a rate of 4 ft/sec. How fast is the height of the shadow changing when the person is 24 feet from the wall? Is the shadow increasing or decreasing in height at this time?



Solution: Let h be the height of the shadow, and d be the distance between the wall and the person. Then we want to find h'. We currently have $d = 24$. We know by similar triangles that $\frac{36-d}{36} = \frac{5}{h}$, which tells us that currently $h = 15$.

Then we have $d' = -4$. We compute

$$\begin{aligned} \frac{-d'}{36} &= \frac{-5h'}{h^2} \\ \frac{1}{9} &= \frac{-h'}{45} \\ h' &= \frac{-45}{9} = -5. \end{aligned}$$

Thus the shadow's height is decreasing by 5 feet per second.