Math 1231 Practice Final Solutions

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- These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
- You will have 120 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, two-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.
- This test has fourteen questions, over eleven pages. **You may not answer all questions.**
  - The first five problems are five pages, representing the five major topics, and you should do all five of them. They are worth twenty points each.
  - The remaining nine problems represent the nine secondary topics. You should select **up to four** of these questions and answer them. Your two best will be worth ten points each; the third and fourth will be worth up to five bonus points. It is better to answer two questions well than three or four questions poorly.
  - If you answer more than four out of the last nine problems, we'll ignore the extra, so please pick and choose.
  - If you perform well on a question on this test it will update your mastery scores. Achieving a 18/20 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

Problem 1 (M1). (a) Compute $\lim_{x\to 25} \dfrac{\sqrt{x}-5}{x-25}$

Solution:

$$\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25} = \lim_{x\to 25} \frac{x-25}{(x-25)(\sqrt{x}+5)} = \lim_{x\to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{10}.$$

(b) Compute $\lim_{x\to 0} \dfrac{\sin(x^2)}{\sin^2(x)}$

Solution: We use the small angle approximation. We rewrite this as

$$\lim_{x\to 0} \frac{\sin(x^2)}{\sin^2(x)} = \lim_{x\to 0} \frac{\sin(x^2)}{x^2} \frac{x}{\sin(x)} \frac{x}{\sin(x)} = 1.$$

(c) Compute $\lim_{x\to -\infty} \dfrac{3x}{\sqrt{4x^2+3}}$

Solution:

$$\lim_{x\to -\infty} \frac{3x}{\sqrt{4x^2+3}} = \lim_{x\to -\infty} \frac{3}{-\sqrt{4+3/x^2}} = \frac{-3}{2}.$$

Problem 2 (M2). (a) Find $\dfrac{d}{dx} \sqrt[4]{\dfrac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4} \cdot \frac{(3x^2-\sin(x^2)2x)(\sin(x^3)+1)-\cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

(b) Find a formula for y' in terms of x and y if $x^8+x^4+y^4+y^6=1$.

Solution:

$$\begin{aligned} 8x^7+4x^3+4y^3\frac{dy}{dx}+6y^5\frac{dy}{dx} &= 0 \\ 8x^7+4x^3 &= (4y^3+6y^5)\frac{dy}{dx} \\ -\frac{4x^7+2x^3}{2y^3+3y^5} &= \frac{dy}{dx}. \end{aligned}$$

Problem 3 (M3). (a) If $f(x)=\sqrt{x}+\tan(\pi x)$, use a linear approximation centered at 4 to estimate $f(4.1)$.

Solution: We have $f'(x)=\frac{1}{2\sqrt{x}}+\pi\sec^2(\pi x)$ so $f'(4)=\frac{1}{4}+\pi$. Then

$$\begin{aligned} f(x) &\approx f(4)+f'(4)(x-4) = 2+0+(\pi+1/4)(x-4) \\ f(4.1) &\approx 2+\frac{\pi}{10}+\frac{1}{40} = \frac{81}{40}+\frac{\pi}{10}. \end{aligned}$$

(b) A curve is defined by the equation $x^4-2x^2y^2+y^4=16$. $(\sqrt{5},1)$. What is the equation of the tangent line to the curve at the point $(\sqrt{5},1)$?

Solution: If we plug in $\sqrt{5}$ for x and 1 for y we get $25-2\cdot 5\cdot 1+1=16$, so the point $(\sqrt{5},1)$ is on the curve.

To find the tangent line, we use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$4x^3 - 4xy^2 = 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

$$\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} = \frac{dy}{dx}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5}\cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4\cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \sqrt{5}(x - \sqrt{5}).$$

Problem 4 (M4). (a) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x-4)(x-1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$f(0) = 7$$

$$f(1) = 14$$

$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$

$$f(5) = 3\cdot 5^4 - 4\cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

(b) Classify the relative extrema of $g(x) = \dfrac{2x-1}{x^2+2}$.

Solution: We have

$$g'(x) = \frac{2(x^2+2) - 2x(2x-1)}{(x^2+2)^2} = \frac{-2x^2 + 2x + 4}{(x^2+2)^2}$$

$$= -2\frac{x^2 - x - 2}{(x^2+2)^2} = -2\frac{(x-2)(x+1)}{(x^2+2)^2}$$

so the critical points are 2 and -1. (The derivative is defined everywhere).

To classify these critical points we need to use either the first or second derivative test. I think the first derivative test looks easier here, purely because I don't want to compute the second derivative. I get the table

	$x-2$	$x+1$	$\frac{-2}{(x^2+2)^2}$	$g'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 2$	$-$	$+$	$-$	$+$
$2 < x$	$+$	$+$	$-$	$-$

Thus we see that there is a relative minimum at -1 and a relative maximum at 2.

But we could use the second derivative test if we really wanted to. We compute

$$g''(x) = -2\frac{(2x-1)(x^2+2)^2 - 2(x^2+2)2x(x^2-x-2)}{(x^2+2)^4}$$
$$g''(-1) = -2\frac{(-3)(3)^2 - 2(3)(-2)(0)}{3^4} = \frac{-2\cdot(-27)}{3^4} = 2/3 > 0$$
$$g''(2) = -2\frac{3(6)^2 - 2(6)4(0)}{6^4} = \frac{-1}{6} < 0.$$

Thus $g''(-1) > 0$ so g has a minimum at -1; and $g''(2) < 0$ so g has a maximum at 2.

Problem 5 (M5)**.** (a) Let $G(x) = \int_1^{x^2+1} t\sqrt{1-t^2}\,dt$. What is $G'(x)$?

Solution: $G'(x) = \sqrt{1-(x^2+1)^2}(x^2+1)\cdot(x^2+1)' = \sqrt{1-(x^2+1)^2}\cdot(x^2+1)\cdot 2x$

(b) Compute $\displaystyle\int \sin^4(t)\cos(t)\,dt$

Solution: We can take $u = \sin(t)$, then we have $du = \cos(t)\,dt$ so we are computing

$$\int u^4\,du = \frac{1}{5}u^5 + C = \frac{\sin^5(t)}{5} + C.$$

(c) **By explicitly changing the bounds of the integral**, compute $\displaystyle\int_0^4 x^3\sqrt{9+x^2}\,dx$.

Solution: Take $u = 9 + x^2$ so that $x^2 = u - 9$ and $dx = u/2x$. Then $u(0) = 9$ and $u(4) = 25$ and we have

$$\begin{aligned}\int_0^4 x^3\sqrt{9+x^2}\,dx &= \int_9^{25} x^2\sqrt{u}\frac{du}{2x} = \int_9^{25}\frac{1}{2}(u-9)\sqrt{u}\,du\\ &= \frac{1}{2}\int_9^{25} u^{3/2} - 9u^{1/2}\,du\\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - 6u^{3/2}\right)\Big|_9^{25} = \frac{1}{2}(1250 - 750 - (486/5 - 162))\\ &= \frac{1}{2}(662 - 486/5) = \frac{3310-486}{10} = \frac{2824}{10} = 1412/5.\end{aligned}$$

Problem 6 (S1)**.** Naming each limit law you use explicitly, compute

$$\lim_{x\to 1}\frac{x^2+4x-5}{x-1}$$

Solution:

$$\begin{aligned}\lim_{x\to1}\frac{x^2+4x-5}{x-1} &= \lim_{x\to1}\frac{(x+5)(x-1)}{x-1} && \text{Arithmetic}\\ &= \lim_{x\to1} x+5 && \text{Almost Identical Functions}\\ &= \lim_{x\to1} x + \lim_{x\to1} 5 && \text{Additivity}\\ &= 1 + \lim_{x\to1} 5 && \text{Identity}\\ &= 1+5 && \text{Constants}\\ &= 6\end{aligned}$$

Problem 7 (S2)**.** Use the Squeeze Theorem to show that $\lim_{x\to5}(x-5)\sin\left(\frac{x^2+1}{x-5}\right) = 0$.

Solution: We have

$$\begin{aligned}
-1 &\le \sin\left(\frac{x^2+1}{x-5}\right) \le 1 \\
-|x-5| &\le (x-5)\sin\left(\frac{x^2+1}{x-5}\right) \le |x-5|.
\end{aligned}$$

We see that $\lim_{x\to 5} -|x-5| = \lim_{x\to 5} |x-5| = 0$, so by the squeeze theorem we know that

$$\lim_{x\to 5}(x-5)\sin\left(\frac{x^2+1}{x-5}\right) = 0.$$

Problem 8 (S3)**. Directly from the definition**, compute $f'(1)$ where $f(x) = \sqrt{x+3}$.
Solution:

$$\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h} \\
&= \lim_{h\to 0}\frac{(4+h)-4}{h(\sqrt{4+h}+\sqrt{4})} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}.
\end{aligned}$$

Problem 9 (S4)**.** Suppose that if a car travels at v miles per hour then its fuel efficiency is $F(v) = 8 + 1.3v - .015v^2$ miles per gallon.

(i) What does the derivative $F'(v)$ represent, and what are its units?

Solution: The derivative $F'(v)$ is the rate at which fuel efficiency increases as your speed increases. The units are miles per gallon per mile per hour, which winds up working out to hours per gallon. (This is a little weird, but it actually makes sense: it's something like how many hours you save by burning an extra gallon of fuel).

(ii) Compute $F'(60)$. What does this tell you?

Solution: $F'(v) = 1.3 - .03v$ hours per gallon so $F'(60) = 1.3 - 1.8 = -.5$ hours per gallon. This tells us that if we are going sixty miles per hour, then increasing our speed by one mile per hour will reduce our gas milage by half a mile per gallon.

Problem 10 (S5)**.** A cone with height h and base radius r has volume $\frac{1}{3}\pi r^2 h$. Suppose we have an inverted conical water tank, of height 4m and radius 6m. Water is leaking out of a small hole at the bottom of the tank. If the current water level is 2m and the water level is dropping at $\frac{1}{9\pi}$ meters per minute, what volume of water leaks out every minute?
Solution: We have $V = \frac{1}{3}\pi r^2 h$ and $r = 3h/2$, and thus

$$\begin{aligned}
V &= \frac{1}{3}\pi(\frac{3h}{2})^2 h = \frac{3}{4}\pi h^3 \\
V' &= \frac{9}{4}\pi h^2 h' \\
V' &= \frac{9}{4}\pi(2)^2\frac{-1}{9\pi} = 1
\end{aligned}$$

So one cubic meter of water is leaking out every minute.

Problem 11 (S6)**.** Let $j(x) = x^4 - 14x^2 + 24x + 6$. We can compute $j'(x) = 4(x+3)(x-1)(x-2)$ and $j''(x) = 4(3x^2 - 7)$. Sketch a graph of j.
Your answer should discuss the domain, asymptotes, limits at infinity, critical points and values, intervals of increase and decrease, and concavity.
Solution: The domain of j is all reals. I'm not going to worry about finding roots now , and there are no obvious symmetries. It's a polynomial of even degree, so it's easy to see that $\lim_{x\to\pm\infty} j(x) = +\infty$.

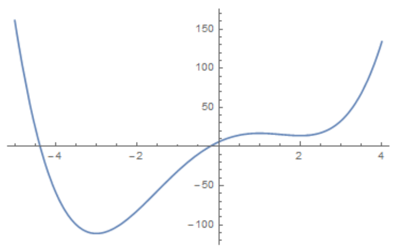
The function j is defined everywhere and is zero at three points. Thus j has three critical points, at $-3, 1, 2$. We compute j at these critical points: $j(-3) = 81 - 126 - 72 + 6 = -111$, $j(1) = 1 - 14 + 24 + 6 = 17$, $j(2) = 14$.

We can make a chart to determine when j increases or decreases:

	$(x+3)$	$(x-1)$	$(x-2)$	$j'(x)$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 2$	$+$	$+$	$-$	$-$
$2 < x$	$+$	$+$	$+$	$+$

So j is increasing between -3 and 1 and when bigger than 2, and j is decreasing when smaller than -3 or between 1 and 2. This implies that j has a relative minimum (of -111) at -3, a relative maximum (of 17) at 1, and a relative minimum of 14 at 2.

$j''(x) = 4(3x^2 - 7)$ is defined eveyrwhere, and is zero when $x^2 = 7/3$, when $x = \pm\sqrt{7/3}$. $j''(x)$ is positive when $|x| > \sqrt{7/3}$ and negative when $|x| < \sqrt{7/3}$.



Graph of $j(x)$

Problem 12 (S7)**.** Use two iterations of Newton's method, starting at 4, to estimate $\sqrt{15}$.

Solution: Set $f(x) = x^2 - 15$, and $x_1 = 4$. We have $f'(x) = 2x$. Then:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 4 - \frac{1}{8} = \frac{31}{8} \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{31}{8} - \frac{31^2/8^2 - 15}{31/4} \\ &= \frac{31}{8} - \frac{1/64}{31/4} = \frac{31}{8} - \frac{1}{31 \cdot 16} \\ &= \frac{31 \cdot 62 - 1}{31 \cdot 16} = \frac{1921}{496} \approx 3.87298. \end{aligned}$$

(You can leave the last number unsimplified on the final.)

Problem 13 (S8)**.** Using **only the definition of Riemann sum** and your knowledge of limits, compute the exact area under the curve $x^2 + x^3$ between $x = 1$ and $x = 3$.

Solution: We compute

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{2}{n} f\left(1 + \frac{2i}{n}\right) = \frac{2}{n} \sum_{i=1}^{n} (1 + 2i/n)^2 + (1 + 2i/n)^3 \\
&= \frac{2}{n} \sum_{i=1}^{n} (1 + 4i/n + 4i^2/n^2) + (1 + 6i/n + 12i^2/n^2 + 8i^3/n^3) \\
&= \frac{2}{n} \sum_{i=1}^{n} 2 + 10i/n + 16i^2/n^2 + 8i^3/n^3 \\
&= \frac{4}{n} \sum_{i=1}^{n} 1 + \frac{20}{n^2} \sum_{i=1}^{n} i + \frac{32}{n^3} \sum_{i=1}^{n} i^2 + \frac{16}{n^4} \sum_{i=1}^{n} i^3 \\
&= \frac{4}{n} \cdot n + \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
\lim_{n \to +\infty} R_n &= \lim_{n \to +\infty} \frac{4}{n} \cdot n + \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\
&= 4 + 10 + \frac{32}{3} + 4 = \frac{86}{3}.
\end{aligned}$$

Problem 14 (S9)**.** What is the centroid (center of mass) of the region bounded by $y = x^3$ and $x = 4$?
Solution:
The area of the region is $\int_0^4 x^3\,dx = \frac{x^4}{4}|_0^4 = 64$. The x coordinate of the center of mass is

$$\overline{x} = \frac{1}{A} \int_0^4 x \cdot x^3\,dx = \frac{1}{64} \cdot \frac{x^5}{5}|_0^4 = \frac{16}{5}$$

and the y-coordinate is

$$\overline{y} = \frac{1}{A} \int_0^4 \frac{1}{2}(x^3)^2\,dx = \frac{1}{128} \frac{x^7}{7}|_0^4 = \frac{128}{7}.$$

Thus the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{16}{5}, \frac{128}{7}\right)$.