Math 1231 Practice Midterm Solutions

Instructor: Jay Daigle

1. These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
2. You will have 75 minutes for this test.
3. You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
4. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.
5. This test has eight questions, over six pages. **You should not answer all eight questions.**
   (a) The first three problems are three pages, representing topics M1, M2, and M3, and you should do all three of them. They are worth twenty points each.
   (b) The remaining five problems represent topics S1 through S5. You should select **up to three** of these questions and answer them. Your two best will be worth ten points each; the third will be worth up to five bonus points. It is better to answer two questions well than three questions poorly.
   (c) If you answer more than three out of the last five problems, we'll ignore the extra, so please pick and choose.
   (d) If you perform well on a question on this test it will update your mastery scores. Achieving a 18/20 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

1		5	
2		6	
3		7	
4		8	
Σ			

Problem 1 (M1)**.** Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(b)

$$\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned}\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3+\sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6})+x/x^3} \\ &= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\ &= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1.\end{aligned}$$

(c)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(d)

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 2 (M2)**.** Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)$

Solution:

$$f'(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \tan\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \frac{\frac{1}{2}(x^2+1)^{-1/2}2x(x+2)-\sqrt{x^2+1}}{(x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4} \cdot \frac{(3x^2-\sin(x^2)2x)(\sin(x^3)+1)-\cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

Problem 3 (M3)**.**

(a) Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$
$$4x^3 - 4xy^2 = 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$
$$\frac{4x^3-4xy^2}{4x^2y-4y^3} = \frac{dy}{dx}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5}\cdot 1^2}{4\sqrt{5}^2\cdot 1 - 4\cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \sqrt{5}(x - \sqrt{5}).$$

(b) Give equation for the linear approximation of the function $f(x) = x\sin(x)$ near the point $a = \pi/2$. Use it to estimate $f(1.5)$.

Solution: We calculate that $f(\pi/2) = \pi/2\sin(\pi/2) = \pi/2$, and $f'(x) = \sin(x) + x\cos(x)$, so $f'(\pi/2) = \sin(\pi/2) + \pi/2\cos(\pi/2) = 1$. So

$$f(x) \approx \pi/2 + 1(x - \pi/2) = x.$$

Thus we have

$$f(1.5) \approx 1.5.$$

(The true answer is $1.49624\ldots$.)

Problem 4 (S1)**.** Explicitly naming each limit law you use, compute

$$\lim_{x\to -1} \frac{\sqrt{x^2+3}}{2x+3} =$$

Solution:

$$\begin{aligned}
\lim_{x\to -1}\frac{\sqrt{x^2+3}}{2x+3} &= \frac{\lim_{x\to-1}\sqrt{x^2+3}}{\lim_{x\to-1}2x+3} && \text{Quotients}\\
&= \frac{\sqrt{\lim_{x\to-1}x^2+3}}{\lim_{x\to-1}2x+3} && \text{Exponents}\\
&= \frac{\sqrt{\lim_{x\to-1}x^2+\lim_{x\to-1}3}}{\lim_{x\to-1}2x+\lim_{x\to-1}3} && \text{Sums}\\
&= \frac{\sqrt{\lim_{x\to-1}x^2+\lim_{x\to-1}3}}{2\lim_{x\to-1}x+\lim_{x\to-1}3} && \text{scalar products}\\
& && \text{[could also use regular product]}\\
&= \frac{\sqrt{(\lim_{x\to-1}x)^2+\lim_{x\to-1}3}}{2\lim_{x\to-1}x+\lim_{x\to-1}3} && \text{Exponents}\\
&= \frac{\sqrt{(\lim_{x\to-1}x)^2+3}}{2\lim_{x\to-1}x+3} && \text{Constants}\\
&= \frac{\sqrt{(-1)^2+3}}{2(-1)+3} = \frac{\sqrt{4}}{1} = 2 && \text{Identity}
\end{aligned}$$

Problem 5 (S2)**.** Show that $\lim_{x\to 0} x\sin\left(\frac{3}{x}\right) = 0$.

Solution: We know that

$$\begin{aligned}
-1 &\le \sin\left(\frac{3}{x}\right) \le 1\\
-|x| &\le x\sin\left(\frac{3}{x}\right) \le |x|
\end{aligned}$$

Since $\lim_{x\to0} -|x| = 0$ and $\lim_{x\to0}|x| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to0} x\sin\left(\frac{3}{x}\right) = 0$.

Problem 6 (S3)**. Directly from the definition of derivative,** compute the derivative of $f(x) = x^2+\sqrt{x}$ at $a = 2$.

Solution:

$$\begin{aligned}
f'(2) &= \lim_{h\to0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to0}\frac{(2+h)^2+\sqrt{2+h}-2^2-\sqrt{2}}{h}\\
&= \left(\lim_{h\to0}\frac{4h+h^2}{h}\right)+\left(\lim_{h\to0}\frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right)\\
&= \left(\lim_{h\to0}4+h\right)+\left(\lim_{h\to0}\frac{1}{\sqrt{2+h}+\sqrt{2}}\right)\\
&= 4+\frac{1}{2\sqrt{2}}.
\end{aligned}$$

Problem 7 (S4)**.** Suppose that $Q(p) = 3p^2+10p-100$ is the number of widgets you can buy at a price of p dollars.

(i) What does the derivative $Q'(p)$ represent, and what are its units?

Solution: The derivative is the rate at which increasing the price increases the number of widgets you can buy (called the marginal elasticity of demand, though you don't need to know that on the test). Its units are widgets per dollar.

(ii) Calculate $Q'(10)$. What does this tell you?

Solution: $Q'(p) = 6p + 10$ so $Q'(10) = 70$. This means that if you are buying widgets for \$10, you can get approximately seventy more widgets if you raise your price ot \$11.

Problem 8 (S5)**.** The surface area of a cube is given by the formula $A = 6s^2$ where s is the length of a side. If the side lengths are increasing by 2 inches per second, how fast is the surface area increasing when the area is 54 square inches?

Solution: We have the data $A = 6s^2, A = 54, s' = 2$. We take a derivative and see that $A' = 12ss'$, so we need to find s. But when $A = 54$ we have

$$\begin{aligned} 54 &= 6s^2 \\ 9 &= s^2 \\ 3 &= s \end{aligned}$$

and thus

$$A' = 12ss' = 12 \cdot 3 \cdot 2 = 72$$

so the area is increasing at 72 square inches per second.