1 Functions and Limits

1.1 Quick Review Facts

Functions

Recall that a *function* is a rule that takes an input and assigns a specific output. Note that a function always gives exactly one output, and always gives the same output for a given input. Here we remember some facts about common functions.

Polynomials: You should remember the quadratic formula, which says that if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is also useful to recall that

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)(a - b) = a^2 - b^2$
- $(a^2 + ab + b^2)(a - b) = a^3 - b^3$.

Rational functions are the ratio of two polynomials.

Trigonometric functions: In this course we will *always* use radians, because they are unitless and thus easier to track (especially when using the chain rule). Useful facts include:

- The most important trigonometric identity, and really the only one you probably need to remember, is $\cos^2(x) + \sin^2(x) = 1$.
- From this you can derive the fact that $1 + \tan^2(x) = \sec^2(x)$.
- $\sin(-x) = -\sin(x)$. We call functions like this "odd".
- $\cos(-x) = \cos(x)$. We call functions like this "even."
- $\sin(x + \pi/2) = \sin(\pi/2 - x) = \cos(x)$
- A fact that we will probably use exactly twice is the sum of angles formula for sine: $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
- Similarly, $\cos(x + y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

Set and interval notation

We write $\{x : \text{condition}\}$ to represent the set of all numbers x that satisfy some condition. We will sometimes write $\mathbb{R}$ to refer to all the real numbers. We will also refer to various intervals:

$(a, b) = \{x : a < x < b\}$	open interval	$[a, b] = \{x : a \leq x \leq b\}$	closed interval
$[a, b) = \{x : a \leq x < b\}$	half-open interval	$(a, b] = \{x : a < x \leq b\}$	half-open interval

1.2 Review of functions

Definition 1.1. A *function* is a rule that takes an input and assigns a specific output. Note that a function always gives exactly one output, and always gives the same output for a given input.

In the abstract, a function can take any type of input and give any type of output. In this class we will primarily study functions whose inputs and outputs are all real numbers.

Definition 1.2. The *domain* of a function is the set of possible valid inputs. The *range* or *image* is the set of possible outputs.

Example 1.3.

1. The function $f(x) = x^2$ has all real numbers in its domain, and its image is the set of non-negative real numbers.
2. The function $f(x) = \sqrt{x}$ has all non-negative real numbers as its domain, and non-negative real numbers as its image.
3. The function $f(x) = \frac{1}{x^2-1}$ has all real numbers except 1 and -1 in its domain, and all real numbers greater than zero or less than or equal to -1 in its image. We can write this set as $\{x : x > 0 \text{ or } x \leq -1\}$, or equivalently as $\{x : x > 0\} \cup \{x : x \leq -1\}$ or $(-\infty, -1] \cup (0, +\infty)$.

Remark 1.4. The word "range" is sometimes used to refer to the type of output a function can have; in this context people also use the word "codomain". In this class we will always use "range" to refer to an output a function can actually produce.

Functions can be described many ways: a verbal description, an algebraic rule, a graph, or a list of possible inputs and the corresponding outputs.

Example 1.5. What are the domain and range of $f(x) = x^3$?

The domain of the function is all real numbers, since we can cube any number. Less obviously, the range is also all reals: if we cube a negative number, we get a negative number, and if we cube a positive number we get a positive number.

Example 1.6. What are the domain and range of $\frac{1}{x-1}$?

The domain is all reals except 1, because we can't divide by zero. (In general, the domain is often "everywhere nothing goes wrong.") The image is all reals except 0, since we can divide 1 by any number except 0 and thus get the reciprocal of any non-zero number.

In other notation, the domain is $\{x : x \neq 1\}$ and the range is $\{x : x \neq 0\}$.

Definition 1.7. A *piecewise function* is a function defined by breaking its domain up into pieces and giving a rule for each piece.

Example 1.8.

1.
$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$
is a piecewise function, given by the rule "If the input is negative, the output is zero; otherwise the output is 1." The domain is all reals and whose range is $\{0, 1\}$.

2.
$$g(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 0 \end{cases}$$
is not a function because it does not give a clear output when given 0 as input.

3.
$$h(x) = \begin{cases} x^2 + 1 & x < 0 \\ 3x - 2 & x > 0 \end{cases}$$
is a piecewise function whose domain does not include 0. The domain is $\{x : x \neq 0\}$ and the range is $(-2, +\infty)$.

4.
$$f(x) = \begin{cases} x + 2 & x \geq 1 \\ x^2 + 2 & x \leq 1 \end{cases}$$
This function might concern you since it appears to have two values for 1; but after looking a bit more closely we see that both pieces define $f(x) = 3$ so we're okay. This is a function whose domain is all reals and whose image is $[2, +\infty)$.

1.2.1 Function Catalog

We will now present a list of functions; we should be familiar with these functions, their graphs, and often their domains and images.

1. A *constant function* is given by $f(x) = c$ for some real number c. It's domain is all real numbers, and its range is the set with one point $\{c\}$.

2. A *linear function* is given by $f(x) = ax + b$. Its domain and range are both all real numbers.

3. A *polynomial function* is given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where n is some positive integer and the a_i are all real numbers. A polynomial is a sum of terms, where each term is some real number multiplied by x raised to a positive integer power.

   The domain of any polynomial is all real numbers.

   (3a) A *quadratic polynomial* is a polynomial whose highest term has exponent 2, given by $f(x) = ax^2 + bx + c$. It has image $\{x : x \geq C\}$ or $\{x : x \leq C\}$ for some real number C.

   It will be useful to recall the quadratic formula; if $f(x) = ax^2 + bx + c$ then $f(x) = 0$ precisely when

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   (3b) A *cubic polynomial* has 3 as its highest exponent, given by $f(x) = ax^3 + bx^2 + cx + d$. Its image is all real numbers.

4. A *rational function* is given by the ratio of two polynomial functions (note the similarity between "ratio" and "rational"). Thus a rational function is of the form

   $$f(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}.$$

   A rational function has domain all real numbers, except for the finite collection of points where the denominator is zero.

   Example 1.9.
   - $f(x) = \frac{x^2+1}{x-1}$ is a rational function with domain $\{x : x \neq 1\}$.
   - $g(x) = \frac{1}{x^4+7}$ is a rational function with domain all reals, since the denominator is never zero for any real number. (The range is $(0, 1/7]$).

5. The function
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases} = \sqrt{x^2}$$
is well-defined since both rules give the same output for 0. This function is called the *absolute value* of x. The piecewise definition is usually more useful. The domain is all reals, and the image is $[0, +\infty)$; in fact, the point of this function is to "sanitize" all your real number inputs into positive numbers.

We will now discuss the exponential functions.

1. The *n-th root function* is given by $f(x) = x^{1/n}$. The number $x^{1/n}$ is the unique positive number y such that $y^n = x$. If n is even then this function has all non-negative numbers in its domain and image; if n is odd then all real numbers are in the domain and image.
2. The *reciprocal function* is given by $f(x) = x^{-1} = \frac{1}{x}$. This function has domain and range $\{x : x \neq 0\}$. It also has the interesting property that $f(f(x)) = x$ for any $x \neq 0$; that is, applying the rule twice gets you back where you started.
3. We can define a general exponential function $f(x) = x^{m/n}$ where m and n are any integers by combining the previous two rules with the rules that
   - $x^a x^b = x^{a+b}$
   - $(x^a)^b = x^{ab}$
   - $x^a y^a = (xy)^a$

Example 1.10. If we wish to calculate $8^{-5/3}$, we can rewrite this as
$$(8^{5/3})^{-1} = ((8^{1/3})^5)^{-1} = (2^5)^{-1} = 32^{-1} = \frac{1}{32}.$$

Example 1.11. Compute $27^{-2/3}$.
$$27^{-2/3} = ((27^{1/3})^2)^{-1} = (3^2)^{-1} = 9^{-1} = \frac{1}{9}.$$

Example 1.12. What is the domain of $f(x) = \dfrac{x^2 - 4}{x^2 + 5x + 6}$?

The domain is all reals except where the denominator is zero. $x^2+5x+6 = (x+2)(x+3)$ is zero when $x = -2$ or $x = -3$, so the domain is $\{x : x \neq -2, -3\}$.

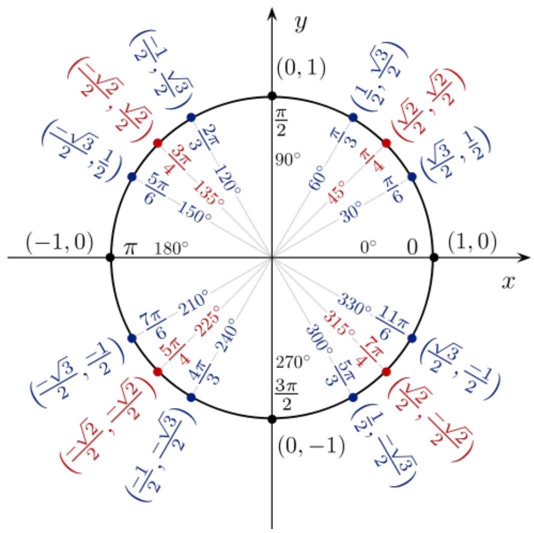


Figure 1.1: The Unit Circle

Now we discuss the trigonometric functions. In calculus we essentially always use radians. Recall that $\sin(x)$ and $\cos(x)$ are given by the unit circle: if we start from the point $(1, 0)$ and rotate x radians counterclockwise, then our x coordinate will be $\cos(x)$ and our y coordinate will be $\sin(x)$. We can also recall that if θ is the measure of a non-right angle of a right triangle, then $\sin(\theta)$ is the ratio of the length of the opposite side to the length of the hypotenuse, and $\cos(\theta)$ is the ratio of the length of the adjacent side to the length of the hypotenuse.

There is one important trigonometric identity we must remember, which is that $\sin^2(x) + \cos^2(x) = 1$; this is just the Pythagorean theorem applied to triangles with hypotenuse of length one.

We can see that sin and cos both have domain all reals, and image $[-1, 1]$.

We also have four other trigonometric functions:

1. $\tan(x) = \frac{\sin(x)}{\cos(x)}$ has domain $\{x : x \neq n\pi + \pi/2\}$ since the function isn't defined when $\cos(x) = 0$, and has image all reals.

2. $\cot(x) = \frac{\cos(x)}{\sin(x)}$ has domain $\{x : x \neq n\pi\}$ since the function isn't defined hwen $\sin(x) = 0$, and has image all reals.

3. $\sec(x) = \frac{1}{\cos(x)}$ has domain $\{x : x \neq n\pi + \pi/2\}$ and image $(-\infty, -1] \cup [1, +\infty)$.

4. $\csc(x) = \frac{1}{\sin(x)}$ has domain $\{x : x \neq n\pi\}$ and image $(-\infty, -1] \cup [1, +\infty)$.

The trigonometric functions also have a few important symmetries:

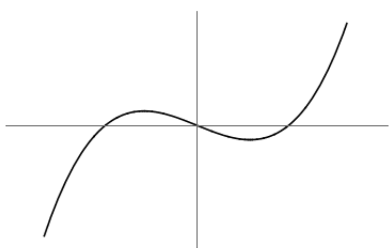
- $\sin(-x) = -\sin(x)$. Functions with this property are called *odd functions*.
- $\cos(-x) = \cos(x)$. Functions with this proprty are called *even functions*.
- $\sin(\pi/2 - x) = \cos(x)$. The sin function is a *reflection* of the cos function around the line $x = \pi/4$.
- $\sin(x + \pi/2) = \cos(x)$. The sin function is a *translation* of the cos function along the x axis.

This leads into our next topic, which is to ask how we can turn some functions into other functions.

1.2.2 Deriving functions from other functions

We can't possibly list every function we will ever use. Instead, let's talk about how to start with a few functions—the ones above—and use them to construct more functions.

Example 1.13. What must I do to graph A to get graph B?



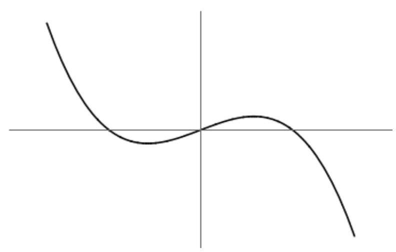
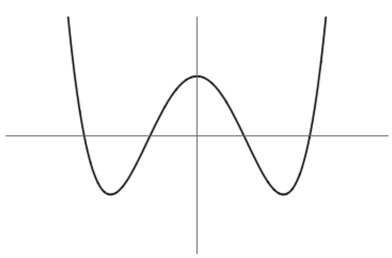


Figure 1.2: Left: graph A, Right: graph B

Example 1.14. What must I do to graph C to get graph D?



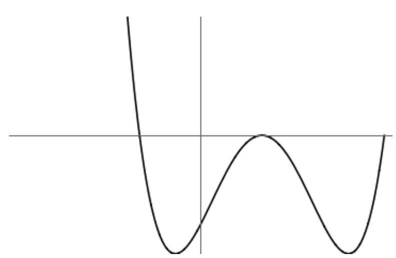


Figure 1.3: Left: graph C, Right: graph D

Now we can move on to the main event: various operations we can apply to a function to get a new function.

Assume that c is a positive real number.

We can *shift* the graph of a function up, down, left, or right:

- The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted up by c units.
- The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted down by c units.
- The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted right by c units.
- The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted left by c units.

Note the perhaps-counterintuitive directions on the last two.

Example 1.15. The first graph is the graph of x^2. What is the second graph?

Figure 1.4: The graphs of x^2 and $x^2 - 1$

Answer: $x^2 - 1$. (Since there's no axis labels, $x^2 - c$ would also be reasonable).

Example 1.16. What do I need to do to the graph of x^3 to get the graph of $(x + 3)^3$?

Figure 1.5: The graphs of x^3 and $(x + 3)^3$

Answer: shift it to the left by three units.

We can also *stretch* the graph of a function vertically or horizontally.

- The graph of $y = c \cdot f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of c. Note c can be less than one here, in which case the graph is shrunk.
- The graph of $y = f(x/c)$ is the graph of $y = f(x)$ stretched horizontally by a factor of c. Note again that c can be less than one, in which case the graph is shrunken.

Example 1.17. If I stretch the function $\sin(x)$ to be twice as tall, what function do I get?

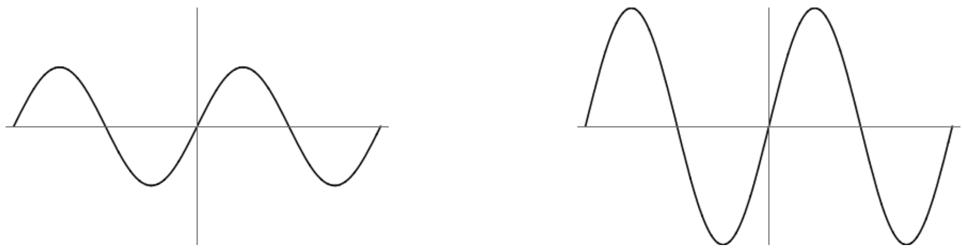


Figure 1.6: The graphs of $\sin(x)$ and $2\sin(x)$

We can also *reflect* a graph about the x axis or y axis (or, with a little creativity, some other axis).

- The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis, that is, flipped top-to-bottom.
- The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis, that is, flipped left-to-right.

Example 1.18. Here is an example of what a function looks like reflected.

Figure 1.7: The graphs of $x^3 + 2x^2$ and $-x^3 + 2x^2$

Figure 1.8: The graphs of $-x^3 - 2x^2$ and $x^3 - 2x^2$

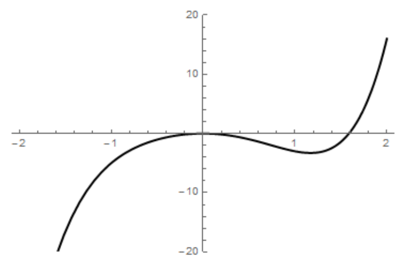


Figure 1.9: The graph of $x^5 - 4x^2$

Example 1.19. Figure **??** is the graph of $x^5 - 4x^2$. What would the graph of $(x+1)^5 - 4(x+1)^2$ look like? What would the graph of $(2x)^5 - 4(2x)^2$ look like?

Figure 1.10: The graphs of $(x+1)^5 - 4(x+1)^2$ and $(2x)^5 - 4(2x)^2$

Example 1.20. Which of the functions $f(x) = x^2+1$, $f(x) = x^3+3$, $f(x) = x^4$, $f(x) = x^5+x$ is even?

Example 1.21. Which of the functions $f(x) = x^2+1$, $f(x) = x^3+3$, $f(x) = x^4$, $f(x) = x^5+x$ is odd?

In general a polynomial with only even-degree terms will be even, and a polynomial with only odd-degree terms is odd. (Hopefully this will be easy to remember!) A polynomial with both even-degree and odd-degree terms is generally neither even nor odd.

Finally, we can combine two functions.

- The function $f+g$ is defined by $(f+g)(x) = f(x)+g(x)$.
- The function $f \cdot g$ is defined by $(f \cdot g)(x) = f(x)g(x)$.
- The function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

This last rule will be very important, and is called *composition of functions*. $f \circ g$ corresponds to putting our input into the function g, and then taking the output and feeding that into the function f. This only makes sense if the image of g is in the domain of f.

Remark 1.22. $f \circ g$ and $g \circ f$ are not the same thing. For instance, if $f(x) = x^2$ and $g(x) = x+1$, then $(f \circ g)(x) = f(x+1) = (x+1)^2 = x^2+2x+1$, but $(g \circ f)(x) = g(x^2) = x^2+1$.

Example 1.23. If $f(x) = \sqrt{x}$ and $g(x) = 3x^2$ then what is$(f \circ g)(x)$? What is the domain? What about $(g \circ f)(x)$?

$(f \circ g)(x) = \sqrt{3x^2}$. This is the same as $\sqrt{3}|x|$. The domain is all reals.

$(g \circ f)(x) = 3\sqrt{x}^2$. This is the same as $3|x|$ but the domain is only $[0, +\infty)$ since we can't plug a negative number into f.

Example 1.24. Can we write x^2+1 as the composition of two simple functions?

Answer: Let $f(x) = x^2$ and $g(x) = x+1$. Then $g(f(x)) = x^2+1$

Can we write $\sqrt{x^3-1}$ as the composition of three simple functions?

Answer: Let $f(x) = x^3$, $g(x) = x-1$, and $h(x) = \sqrt{x}$. Then $h(g(f(x))) = \sqrt{x^3-1}$.

1.3 Informal Continuity and Limits

Let's start with an easy question:

Question 1.25. What is the square root of four?

Everyone can probably tell me that the answer is "two". So now let's do a harder one:

Question 1.26. What is the square root of five?

Without a calculator, you probably can't tell me the answer. But you should be able to make a pretty good guess. Five close to four; so $\sqrt{5}$ should be close to two.

We call this sort of estimate a *zeroth-order approximation.* In a zeroth-order approximation, we only get to use one piece of information: the value of our function at a specific number. Then we use that information to estimate its value at nearby numbers.

We can only do so good a job with that limited amount of information, but we can still do a surprising amount.

Example 1.27. Suppose $f(1) = 36, f(2) = 35, f(3) = 38, f(4) = 38$. What can we say to estimate $f(5)$?

From looking at the data we have, it seems like $f(5)$ should be 38 or 39, probably. But it's actually 45. These are the low temperatures in Pasadena for the first five days of this year.

Often tomorrow's temperature will be similar to today's temperature. But there's no guarantee.

This example shows that we can't always do what we did with $\sqrt{5}$. Some functions jump around too much for this sort of approximation thing to work; values of similar inputs don't have similar outputs.

We don't like these functions, precisely because they're hard to think about or understand. So we're mostly going to look at functions that we *can* approximate effectively.

Definition 1.28 (Informal)**.** We say a function f is *continuous* at a number a if whenever x is close to a, then $f(x)$ is close to $f(a)$.

In other words, for a continuous function, when x and a are close together, then $f(a)$ is a decent approximation for $f(x)$.

Another way to think of this is that the function f is continuous at a if it doesn't "jump" at a.

There are a few different ways for a function to not be continuous at a given number. I will categorize these more carefully in a couple days, but right now I want to show you a few different things that can happen.

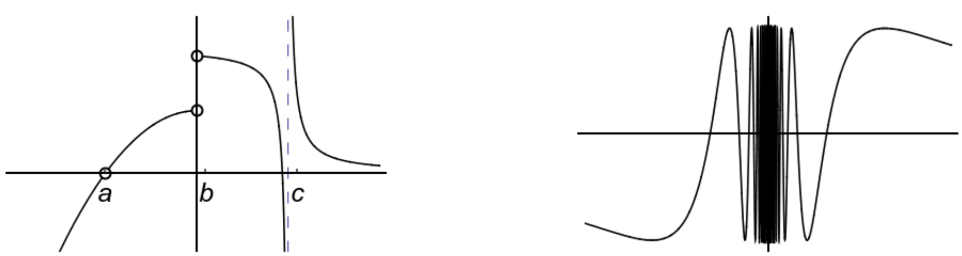


Figure 1.11: Left: a: removable discontinuity; b: jump discontinuity; c: infinite discontinuity. Right: bad discontinuity

Some functions get even worse than that. My two favorite discontinuous functions are:

$$T(x) = \begin{cases} 1/q & x = p/q \text{ rational} \\ 0 & x \text{ irrational} \end{cases} \qquad \chi(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

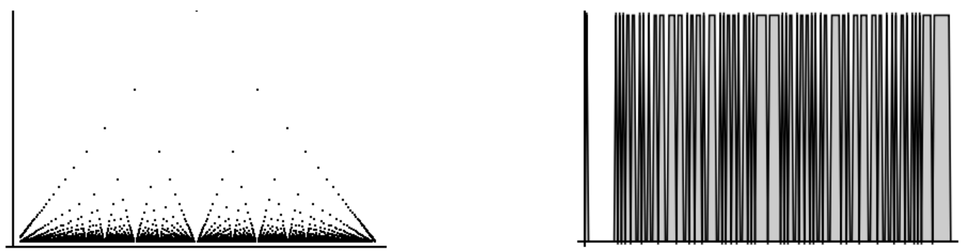


Figure 1.12: Left: $T(x)$ is really discontinuous. Right: $\chi(x)$ is really really discontinuous

In fact, in some sense "most functions" aren't at all continuous. If you found away to choose $f(x)$ completely at random for each real number x, you would get a spectacularly discontinuous function. But you would never actually be able to describe it sensibly.

But for the most part this isn't a problem. Most of the functions that we can easily describe are continuous most of the time. And so when approximating functions we don't understand, we often assume it's reasonably continuous.

Fact 1.29. *Any reasonable function given by a reasonable single formula is continuous at any number for which it is defined.*

In particular, any function composed of algebraic operations, polynomials, exponents, and trigonometric functions is continuous at every number in its domain.

If a function is continuous at every number in its domain, we just say that it is continuous. Note, importantly, that a continuous function doesn't have to be continuous at every real number.

Example 1.30. The function

$$f(x) = \frac{x^3 - 5x + 1}{(x-1)(x-2)(x-3)}$$

is "reasonable", so it is continuous. This means that it is continuous exactly on its domain, which is $\{x : x \neq 1, 2, 3\}$.

Example 1.31. Where is $\sqrt{1 + x^3}$ continuous?

Answer: Root functions are continuous on their domains. $1 + x^3 \geq 0$ when $x \geq -1$ so the function is continuous on its domain, $[-1, +\infty)$.

Remark 1.32. Sometimes we might also talk about functions that are "continuous from the right" at a. This means that $f(a)$ is a good approximation of $f(x)$ if x is close to a and also bigger than—and thus to the right of—a.

In order to understand continuity better, it's helpful to turn the question around and look at things from the opposite direction. (This is a trick that's often useful in math). So instead of asking whether we can estimate $f(x)$ given $f(a)$, we'll turn this around. If we know $f(x)$ for every x near a, what can we say about $f(a)$?

Definition 1.33. Suppose a is a real number, and f is a function which is defined for all x "near" the number a. We say "The *limit* of $f(x)$ as x approaches a is L," and we write

$$\lim_{x \to a} f(x) = L,$$

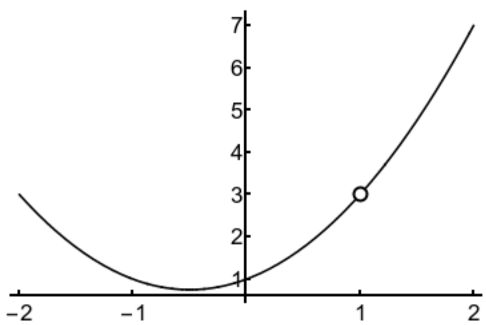
if we can make $f(x)$ get as close as we want to L by picking x that are very close to a.

Graphically, this means that if the x coordinate is near a then the y coordinate is near L. Pictorially, if you draw a small enough circle around the point $(a, 0)$ on the x-axis and look at the points of the graph above and below it, you can force all those points to be close to L.

Notice that we're trying to use knowing $f(x)$ to tell us what happens near a. So we specifically ignore the value of $f(a)$ even if we already know it.

Example 1.34. Let's consider the function $f(x) = \frac{x^3-1}{x-1}$. We can see the graph below. Notice that the function isn't defined at $a = 1$, so $f(1)$ is meaningless and we can't compute it.

But f is defined for all x near 1, so we can compute the limit. Looking at the graph and estimating suggests that when x gets close to 1, then $f(x)$ gets close to 3, and so we can say that $\lim_{x \to 1} f(x) = 3$.



That last example worked, but we basically just eyeballed it. We want a way to actually justify our claims. We can do that using two core principles. The first is what I call the Almost Identical Functions property.

Lemma 1.35 (Almost Identical Functions)**.** *If $f(x) = g(x)$ on some open interval $(a-d, a+d)$ surrounding a, except possibly at a, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ whenever one limit exists.*

This tells us that two functions have the same limit at a if they have the same values near a. This makes sense, because the limit only depends on the values near a.

How does this help us? Ideally, we take a complicated function and replace it with a simpler function.

Example 1.36. Above, we looked at the function $f(x) = \frac{x^3-1}{x-1}$. You may know that we can factor the numerator; thus we in fact have $f(x) = \frac{(x-1)(x^2+x+1)}{x-1}$.

At this point you probably want to cancel the $x-1$ term on the top and the bottom. But in fact that would change the function! For $f(1)$ isn't defined. But the function $g(x) = x^2+x+1$ is perfectly well-defined at $a = 1$. Thus $f(1) \neq g(1)$, and so f and g can't be the same function.

However, they do give the same value if we plug in any number other than 1. If $y \neq 1$ then $y - 1 \neq 0$, so we have

$$f(y) = \frac{(y-1)(y^2+y+1)}{y-1} = y^2 + y + 1 = g(y).$$

Thus f and g aren't the same, but they are *almost* the same. So lemma **??** tells us that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$.

However, this doesn't actually do everything we want it to do. We've replaced a complicated function $f(x) = \frac{x^3-1}{x-1}$ with a simpler function $g(x) = x^2 + x + 1$, but we still haven't figured out what to do with that function.

This leads to our second principle. We started off talking about continuous functions, and said that if f is continuous at a, then $f(a)$ is a good estimate for $f(x)$ when x is near to a. In other words, when x is near a then $f(x)$ is near $f(a)$—so $\lim_{x\to a} f(x) = f(a)$.

This really is the same as the less formal definition we gave at the beginning of this section. There, we said that f is continuous if $f(a)$ is a good approximation for $f(x)$; here we say that f is continuous if $f(x)$ is a good approximation for $f(a)$. This also clarifies *how good* the approximation needs to be. For f to be continuous, the approximation needs to get perfect as x gets close to a.

Example 1.37. The *Heaviside Function* or *step function* is given by

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

It is often used in electrical engineering applications to describe the current running through a switch before and after it has been flipped.

We can ask: what is $\lim_{x\to 0} H(x)$?

There isn't one: no matter how close x gets to 0, sometimes $H(x)$ will be 0 and sometimes it will be 1. So there is no one value that approximates $H(x)$ for any x near a.

However, the Heaviside function clearly behaves well if look only at one side or the other of it. And just as we could talk about continuity to one side or the other, we can talk about *one-sided limits.*

Definition 1.38. Suppose a is a real number, and f is a function which is defined for all $x < a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the left is L," and we write

$$\lim_{x\to a^-} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but less than) a.

Suppose a is a real number, and f is a function which is defined for all $x > a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the right is L," and we write

$$\lim_{x\to a^+} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but greater than) a.

Under this definition, we see that $\lim_{x\to 0^-} H(x) = 0$ and $\lim_{x\to 0^+} H(x) = 1$.

Example 1.39. What is $\lim_{x\to 1^-} f(x)$ if $f(x) = \begin{cases} x^2+2 & x > 1 \\ x-3 & x < 1 \end{cases}$?

Answer: -2.

1.4 A Formal Definition of Limits

1.4.1 The $\epsilon - \delta$ definition

We start by giving a rigorous, formal, and intimidating-looking definition of a limit.

Definition 1.40. Suppose a is a real number, and f is a function defined on some open interval containing a, except possibly for at a. We say the *limit* of $f(x)$ as x approaches a is L, and write

$$\lim_{x\to a} f(x) = L,$$

if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$.

This looks scary, but you should notice that this is *exactly the same thing we said before* in Definition **??**. The letter ϵ represents "how close we want $f(x)$ to get to L" and δ represents "how close x needs to get to a".

Then this definition says that if we pick any margin of error $\epsilon > 0$, then there is some distance δ such that if x is within distance δ of a, then $f(x)$ is within our margin of error ϵ of L.

Remark 1.41. The Greek letter epsilon (ϵ) became the letter "e", and stands for "error". The Greek letter delta (δ) became the letter "d", and stands for "distance". This isn't just a mnemonic for you; this is actually why those letters were chosen.

Example 1.42.

1. If $f(x) = 3x$ then prove $\lim_{x\to 1} f(x) = 3$.

   Let $\epsilon > 0$ and set $\delta = \underline{\epsilon/3}$. Then if $|x-1| < \delta$ then

   $$|f(x) - 3| = |3x - 3| = 3|x-1| < 3\delta = \epsilon.$$

2. If $f(x) = x^2$ then prove $\lim_{x\to 0} f(x) = 0$.

   Let $\epsilon > 0$ and set $\delta = \underline{\sqrt{\epsilon}}$. Then if $|x - 0| < \delta$, then

   $$|f(x) - 0| = |x^2| = |x|^2 < (\sqrt{\epsilon})^2 = \epsilon.$$

3. If $f(x) = \frac{x^2-1}{x-1}$ then $\lim_{x\to 1} f(x) = 2$.

   This is harder to see at first, until we recall or notice that this function is mostly the same as $x + 1$.

   Let $\epsilon > 0$ and let $\delta = \underline{\epsilon}$. Then if $0 < |x-1| < \delta$, we have

   $$\begin{aligned} |f(x) - 2| &= \left| \frac{x^2-1}{x-1} - 2 \right| \\ &= |x + 1 - 2| && \text{since } x \neq 1 \\ &= |x - 1| < \delta = \epsilon. \end{aligned}$$

Remark 1.43. Despite the fact that we set δ as the first thing we do in the proof, we often figure out what it should be last. I strongly recommend beginning your proof by writing "And set $\delta =$ ____" and then working out the proof. By the time you get to the end you'll know what δ needs to be and you can go back and fill in th blank.

Example 1.44. If $f(x) = 4x - 2$ then find (with proof!) $\lim_{x\to -2} f(x)$.

We first need to generate a "guess". This is a nice function, so it seems like the answer should be close to $f(-2) = -10$.

Let $\epsilon > 0$ and set $\delta = \underline{\epsilon/4}$. Then if $|x - (-2)| < \delta$ we compute

$$|f(x) + 10| = |4x - 2 + 10| = |4x + 8| = 4|x + 2| < 4\delta = \epsilon.$$

Example 1.45. If $f(x) = x^2$ find (with proof!) $\lim_{x \to 3} f(x)$.

We first need to generate a "guess". This is a nice, should-be-continuous function, so it seems like the answer should be close to $f(3) = 9$.

Let $\epsilon > 0$ and set $\delta \leq \underline{\epsilon/7, 1}$. Then if $|x - 3| < \delta$ we compute

$$|x^2 - 9| = |x + 3| \cdot |x - 3| < |x + 3|\delta$$

but this is kind of a problem because we still have an x floating around. But logically, we know that if δ is small enough, x will be close to 3 and thus $|x + 3|$ will be close to 6.

To guarantee that $|x + 3|$ is actually close to 6, we'll require $\delta \leq 1$ as well. Then we compute

$$\begin{aligned} |x^2 - 9| &< |x + 3|\delta = |(x - 3) + 6| \cdot \delta \\ &\leq (|x - 3| + |6|)\,\delta && \text{by the triangle inequality} \\ &< (1 + 6)\delta = 7\delta. \end{aligned}$$

Notice we said that $|x + 3|$ would be close to 6, and what we actually showed is that $|x + 3| \leq 7$–which of course it is if it is close to 6.

So now we just need to make sure δ is small enough that $7\delta \leq \epsilon$, so in addition to letting $\delta \leq 1$ we also let $\delta \leq \epsilon/7$, so we have

$$|x^2 - 9| < 7\delta = 7\epsilon/7 = \epsilon.$$

Remark 1.46.

- We often use an approach of isolating all our xs and turning them into an $x - 3$ or $x - a$ or whatever we *know how to control*. Since in example **??** we know that $|x - 3| < \delta$ we want to turn all our xs into $|x - 3|$s. Then we can deal with whatever is left over.

- Notice that here we didn't actually say what δ is; we just listed some properties it needs to have, by saying that $\delta \leq \epsilon/12, 1$. If we want to pick out a specific number, we can write $\delta = \min(\epsilon/12, 1)$, but this isn't actually necessary.

Example 1.47. If $f(x) = x^2 + x$, find (with proof) $\lim_{x\to 2} f(x)$.

This is a continuous function, so it seems like the answer should be close to $f(2) = 6$.

Let $\epsilon > 0$ and set $\delta < \sqrt{\epsilon/2, \epsilon/10}$. Then if $0 < |x-2| < \delta$ we have

$$\begin{aligned}
|f(x) - 6| &= |x^2 + x - 6| = |(x^2 - 4) + (x-2)| \\
&\leq |x^2 - 4| + |x - 2| && \text{(triangle inequality)} \\
&= |x-2| \cdot |x+2| + |x-2| = |x-2|\left(|x+2| + 1\right) \\
&= |x-2|\left(|x-2+4| + 1\right) \leq |x-2|\left(|x-2| + 5\right) && \text{(triangle inequality)} \\
&< \delta(\delta + 5) = \delta^2 + 5\delta.
\end{aligned}$$

You could try to figure out exactly when $\delta^2 + 5\delta = \epsilon$, and after some quadratic formula-ing you'd find you need $\delta \leq \frac{-5+\sqrt{25+4\epsilon}}{2}$. But that's tedious and actually way too much work. (But if you prefer this approach it's perfectly acceptable).

It's easier to instead list two conditions: we let $\delta \leq \sqrt{\epsilon/2}, \epsilon/10$. Then $\delta^2 \leq \epsilon/2$ and $5\delta \leq \epsilon/2$, and we have

$$|f(x) - 6| < \delta^2 + 5\delta \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Example 1.48. Now suppose

$$g(x) = \begin{cases} x^2 + x & x \neq 2 \\ 0 & x = 2 \end{cases}$$

What is $\lim_{x\to 2} g(x)$?

This looks really nasty, but is actually easy after we already did Example **??**.

The limit doesn't care about what happens at any one specific point, and especially doesn't care about what happens at 2. So for our purposes, this function is the same as $f(x) = x^2 + x$, and thus the limit is, as before, 6.

Let $\epsilon > 0$, and let $\delta < \sqrt{\epsilon/2}, \epsilon/10$. Then if $0 < |x-2| < \delta$ we have

$$|g(x) - 6| = \left|x^2 + x - 6\right| < \epsilon$$

as computed in Example **??**. (This is a completely valid proof as written!)

1.4.2 Limit Laws

We now hopefully have a good understanding of what we want limits to *mean*. But this sort of proof process would be super cumbersome if we needed to use it every time we wanted to compute a limit. Fortunately, we can make things much simpler. In this (sub)section we'll introduce basic ideas that we use to make computing limits reasonable; in the next couple of sections we'll see how we do this in practice.

Our approach to computing limits begins with three basic principles, the most important of which we've already seen.

Lemma 1.49 (Identity). *Let a be a real number. Then $\lim_{x\to a} x = a$.*

Proof. Let $\epsilon > 0$ and let $\delta = \epsilon$. If $|x-a| < \delta$, then $|x-a| < \delta = \epsilon$. □

Lemma 1.50 (Constants). *Prove that if a, c are real numbers, then $\lim_{x\to a} c = c$.*

Proof. Let $\epsilon > 0$, and set $\delta = 1$. Then if $0 < |x-a| < \delta$ we have $|f(x) - c| = |c - c| = 0 < \epsilon$. □

Lemma 1.51 (Almost Identical Functions). *If $f(x) = g(x)$ on some open interval $(a-d, a+d)$ surrounding a, except possibly at a, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ whenever one limit exists.*

Proof. Suppose $\lim_{x\to a} f(x) = L$. Let $\epsilon > 0$; then there is some δ_1 such that if $0 < |x-a| < \delta_1$ then $|f(x) - L| < \epsilon$. Then let $\delta < d, \delta_1$. If $0 < |x-a| < \delta$ then $g(x) = f(x)$, and thus

$$|g(x) - L| = |f(x) - L| < \epsilon.$$

□

But by themselves, these results aren't terribly interesting; all of those functions are boring! But importantly, we can also learn how limits interact with basic algebraic operations, which allows us to break complicated expressions up into these simple parts.

Proposition 1.52. *Suppose c is a constant real number, and f and g are functions such that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ exist. Then*

1. *(Additivity)* $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.

   Proof. Let $\epsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f(x) - L_1| < \epsilon/2$, and if $0 < |x-a| < \delta_2$ then $|g(x) - L_2| < \epsilon/2$.

   Let $\delta \leq \delta_1, \delta_2$. Then if $0 < |x-a| < \delta$, we compute

   $$|f(x)+g(x)-(L_1+L_2)| = |(f(x)-L_1)+(g(x)-L_2)| \leq |f(x)-L_1|+|g(x)-L_2| < \epsilon/2+\epsilon/2 = \epsilon.$$

   □

2. *(Scalar multiples)* $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$

Proof. If $c = 0$ then the left hand side is $\lim_{x\to a} 0 = 0$ and the right hand side is $0L_1 = 0$ so the equality holds.

If $c \neq 0$, then let $\epsilon > 0$. Then by definition of limit, there exists some δ so that if $0 < |x-a| < \delta$ then $|f(x) - L_1| < \epsilon/c$.

Then if $0 < |x-a| < \delta$, we have

$$|cf(x) - cL_1| = c|f(x) - L_1| < c(\epsilon/c) = \epsilon,$$

which is what we wanted to show. □

3. *(Products)* $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

Proof. Let $\epsilon > 0$. Then there exist δ_1, δ_2 such that

- if $0 < |x-a| < \delta_1$ then $|f(x) - L_1| < \epsilon/(2|L_2|), 1$,
- and if $0 < |x-a| < \delta_2$ then $|g(x) - L_2| < \epsilon/(2|L_1| + 2)$.

Set $\delta \leq \delta_1, \delta_2$. Then if $0 < |x-a| < \delta$, we compute

$$\begin{aligned}
|f(x)g(x) - L_1L_2| &= |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1L_2| \\
&\leq |f(x)g(x) - f(x)L_2| + |f(x)L_2 - L_1L_2| \\
&= |f(x)| \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\
&= |f(x) - L_1 + L_1| \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\
&\leq (|f(x) - L_1| + |L_1|) \cdot |g(x) - L_2| + |L_2| \cdot |f(x) - L_1| \\
&< (1 + |L_1|)\left(\epsilon/(2|L_1| + 2)\right) + |L_2| \cdot \epsilon/(2Ł_2|) \\
&= \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

□

4. *(Quotients) That last rule also works with division if that makes sense: if* $\lim_{x\to a} g(x) \neq 0$*, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

Proof. I'm not going to prove this because it's really long and annoying and not very informative. It's a lot like the last proof except more tedious. If you're feeling masochistic you can probably prove it yourself. □

5. *(Exponents) The rule for multiplication extends to exponentials:* $\lim_{x\to a}(f(x)^n) = (\lim_{x\to a} f(x))^n$. *Also roots:* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, *assuming all the functions make sense.*

   Proof. We're only going to prove this for the case of $f(x)^n$ where n is a positive integer. The other proofs are basically the same, but this has less bookkeeping.

$$\begin{aligned}
\lim_{x\to a} f(x)^n &= \lim_{x\to a} f(x)\cdot f(x)^{n-1} \\
&= \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} f(x)^{n-1}\right) && \text{by the rule on products} \\
&\vdots \\
&= \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} f(x)\right)\cdot\dots\cdot\left(\lim_{x\to a} f(x)\right) \\
&= \left(\lim_{x\to a} f(x)\right)^n
\end{aligned}$$

   Formally we should write this up as a "proof by induction", which you can learn about in Math 2971. □

Example 1.53. 1.

$$\begin{aligned}
\lim_{x\to 1} x^3 &= \left(\lim_{x\to 1} x\right)^3 && \text{Exponents} \\
&= 1^3 && \text{Identity} \\
&= 1
\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x\to 1}(x+1)^3 - 2 &= \lim_{x\to 1}(x+1)^3 - \lim_{x\to 1} 2 && \text{Additivity} \\
&= \left(\lim_{x\to 1}(x+1)\right)^3 - 2 && \text{Exponents and Constants} \\
&= \left(\lim_{x\to 1} x + \lim_{x\to 1} 1\right)^3 - 2 && \text{Additivity} \\
&= (1+1)^3 - 2 && \text{Identity and Constants} \\
&= 2^3 - 2 = 8 - 2 = 6.
\end{aligned}$$

3.

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2}{x} &= \frac{\lim_{x\to 1} x^2}{\lim_{x\to 1} x} && \text{Quotients}\\
&= \frac{\left(\lim_{x\to 1} x\right)^2}{\lim_{x\to 1} x} && \text{Exponents}\\
&= \frac{1^2}{1} && \text{Identity}\\
&= 1/1 = 1.
\end{aligned}$$

We can also approach this problem a different way, since this function is just the same as x everywhere except at 0:

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2}{x} &= \lim_{x\to 1} x && \text{Almost Identical Functions}\\
&= 1 && \text{Identity}
\end{aligned}$$

4.

$$\begin{aligned}
\lim_{x\to 0}\frac{x^2}{x} &= \lim_{x\to 0} x && \text{Almost Identical Functions}\\
&= 0
\end{aligned}$$

Unlike the previous problem, we *cannot* use the Quotient property here because the bottom approaches zero. Compare:

$$\begin{aligned}
\lim_{x\to 0}\frac{x}{x^2} &= \lim_{x\to 0}\frac{1}{x} && \text{Almost Identical Functions}\\
&\neq \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} x}
\end{aligned}$$

The last step doesn't work because now we're dividing by zero, which we can never do. This limit is in fact $\pm\infty$, and we'll look at how to show that without a proof from the definition soon.

Of course, even showing all these steps gets tedious, so you don't have to do that unless I explicitly ask you to. (However, it will be a topic on a mastery quiz.) It's useful to be able to do this when you want to check your work carefully, or when you're working with something particularly tricky.

1.5 Continuity and Computing Limits

Now that we understand limits, we can return to continuity.

Definition 1.54 (Formal)**.** We say that f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

This definition works in both directions. If we want to know whether a function is continuous, we can check its limits; and if we want to know the limit of a continuous function, we can find it by plugging in.

This really is the same as the less formal definition we gave in section **??**. There, we said that f is continuous if $f(a)$ is a good approximation for $f(x)$; here we say that f is continuous if $f(x)$ is a good approximation for $f(a)$. This also clarifies *how good* the approximation needs to be. For f to be continuous, the approximation needs to get perfect as x gets close to a.

The definition of continuity says that $\lim_{x \to a} f(x) = f(a)$. This secretly actually requires three distinct things to happen:

1. The function is defined at a; that is, a is in the domain of f.

2. $\lim_{x \to a} f(x)$ exists.

3. The two numbers are the same.

There are a few different ways for a function to be discontinuous at a point:

1. A function f has a *removable discontinuity* at a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.

2. A function f has a *jump discontinuity* at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are unequal.

3. A function f has a *infinite discontinuity* if f takes on aribtrarily large or small values near a. We'll talk about this more soon.

4. It's also possible for the one-sided limits to not exist, but this doesn't have a special name. We'll see this with $\sin(1/x)$ when we study trigonometric functions in section **??**. In this class, I'll just call a function like this *really bad*. But we'll mostly avoid talking about them.

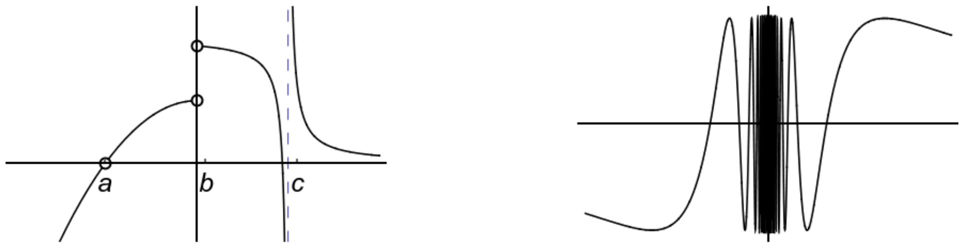


Figure 1.13: We saw this picture in section **??**, but now we have language to talk about it.

A common informal definition is that a continuous function is one whose we can draw without lifting our pencil from the paper. Once we make this precise, this is another way to think about continuous functions. And we make it precise via the Intermediate Value Theorem

Theorem 1.55 (Intermediate Value Theorem)**.** *Suppose f is continuous (and defined!) on the closed interval $[a, b]$ and y is any number between $f(a)$ and $f(b)$. Then there is a c in (a, b) with $f(c) = y$.*

Example 1.56. Suppose $f(x)$ is a continuous function with $f(0) = 3, f(2) = 7$. Then by the Intermediate Value Theorem there is a number c in $(0, 2)$ with $f(c) = 5$.

Example 1.57. Let $g(x) = x^3 - x + 1$. Use the Intermediate Value Theorem to show that there is a number c such that $g(c) = 4$.

To use the intermediate value theorem, we need to check that our function is continuous, and then find one input whose output is less than 4, and another whose output is greater than 4. g is a polynomial and thus continuous. Testing a few values, we see $g(0) = 1, g(1) = 1, g(2) = 7$. Since $g(1) = 1 < 4 < 7 = g(2)$, by the Intermediate Value Theorem ther is a c in $(1, 2)$ with $g(c) = 4$.

Example 1.58. Show that there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

We know that sin is a continuous function, and that $\sin(0) = 0$ and $\sin(\pi/2) = 1$. Since $0 < 1/3 < 1$, by the Intermediate Value Theorem there is a θ in $(0, \pi/2)$ such that $\sin(\theta) = 1/3$.

Remark 1.59*.* The converse of this theorem is not true. It is possible to have a function that satisfies the conclusions of the Intermediate Value Theorem, but is not continuous; these functions are called Darboux Functions.

For example, let $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then f satisfies the conclusion of the intermediate value theorem: it's continuous except at zero, so the theorem works on any interval that doesn't contain zero. Any interval containing zero contains every value in $[-1,1]$, so if $a < 0 < b$ and y is between $f(a)$ and $f(b)$, then $-1 \leq y \leq 1$ and so there is a c in (a,b) such that $f(c) = y$. Thus f is Darboux.

Historically, the main reason we didn't take this as the definition of continuous, instead of the limit definition that we actually use, is that we didn't want to treat functions like this as "continuous".

1.5.1 Limits of Continuous Funtions

This definition does a few things for us:

1. It gives us a clear rule for when a function is continuous. In particular, it will resolve questions about edge-case "weird" functions like $\sin(1/x)$, as we'll discuss in section **??**.

2. If we know a function is continuous, we can easily compute its limit just by plugging in the value.

3. The conclusion of our discussion of limit laws in section **??** is that when functions are made up of algebraic operations, they are continuous whenever they are defined.

Example 1.60. 1. The function $f(x) = 3x$ is continuous at 1, so $\lim_{x\to 1} f(x) = f(1) = 3$.

2. The function $f(x) = x^2$ is continuous at 0, so $\lim_{x\to 0} f(x) = f(0) = 0$.

3. The function $f(x) = \frac{x^2-1}{x-1}$ is definitely not continuous at 1, because it's not defined there. But we can use almost identical functions:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2.$$

Example 1.61. If $f(x) = \frac{x-1}{x^2-1}$ then what is $\lim_{x\to 1} f(x)$?

Answer: 1/2. If $x \neq 1$, then

$$f(x) = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}.$$

We know that $\frac{1}{x+1}$ is continuous, and that it is defined at $a = 1$. Thus $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

Example 1.62. $\lim_{x\to -2} \frac{(x+1)^2-1}{x+2} = \lim_{x\to -2} \frac{x^2+2x+1-1}{x+1} = \lim_{x\to -2} \frac{x(x+2)}{x+2} = \lim_{x\to -2} x = -2$.

Note that $\frac{x(x+2)}{x+2} \neq x$, but their limits at 0 are the same because the functions are the same near 0 (and in fact everywhere except at 0).

Example 1.63. What is $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$?

We use a trick called multiplication by the conjugate, which takes advantage of the fact that $(a+b)(a-b) = a^2 - b^2$. This trick is used *very often* so you should get comfortable with it.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} &= \lim_{x\to 0} \frac{\sqrt{9+x}-9}{x} \frac{\sqrt{9+x}+3}{\sqrt{9+x}+3} \\
&= \lim_{x\to 0} \frac{(9+x)-3}{x(\sqrt{9+x}+3)} = \lim_{x\to 0} \frac{x}{x(\sqrt{9+x}+3)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{9+x}+3} = \frac{1}{\lim_{x\to 0} \sqrt{9+x}+3} = \frac{1}{6}.
\end{aligned}$$

Example 1.64. What is $\lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2}$?

$$\begin{aligned}
\lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} &= \lim_{x\to 1} \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\
&= \lim_{x\to 1} \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)} \\
&= \lim_{x\to 1} -(\sqrt{5-x}+2) = -4.
\end{aligned}$$

Example 1.65. The *Heaviside Function* or *step function* is given by

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

It is often used in electrical engineering applications to describe the current running through a switch before and after it has been flipped.

We can ask: what is $\lim_{x\to 0} H(x)$?

There isn't one: no matter how close x gets to 0, sometimes $H(x)$ will be 0 and sometimes it will be 1. So there is no one value that approximates $H(x)$ for any x near a.

However, the Heaviside function clearly behaves well if look only at one side or the other of it. And just as we could talk about continuity to one side or the other, we can talk about *one-sided limits.*

Definition 1.66. Suppose a is a real number, and f is a function which is defined for all $x < a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the left is L," and we write

$$\lim_{x\to a^-} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but less than) a.

Suppose a is a real number, and f is a function which is defined for all $x > a$ that are "near" the number a. We say "The limit of $f(x)$ as x approaches a from the right is L," and we write

$$\lim_{x \to a^+} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to (but greater than) a.

Under this definition, we see that $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^+} H(x) = 1$.

Example 1.67. What is $\lim_{x \to 1^-} f(x)$ if $f(x) = \begin{cases} x^2 + 2 & x > 1 \\ x - 3 & x < 1 \end{cases}$?

Answer: -2.

Example 1.68. The Heaviside function of example **??** is not continuous, since there's a jump at 0.

It is continuous from the right at 0, since $\lim_{x \to 0^+} H(x) = 1 = H(0)$. This function is not continuous from the left, since $\lim_{x \to 0^-} H(x) = 0 \neq H(0)$.

Definition 1.69. A function is *continuous from the right at a* if $\lim_{x \to a^+} f(x) = f(a)$.

A function is *continuous from the left at a* if $\lim_{x \to a^-} f(x) = f(a)$.

Proposition 1.70. *A function is continuous at a if and only if it is continuous from the left and from the right at a.*

Remark 1.71. At a jump discontinuity, a function will often be continuous from one side but not the other. This is not necessarily the case, though: consider the function

$$f(x) = \begin{cases} 2 & x > 0 \\ 1 & x = 0 \\ 0 & x < 0 \end{cases}$$

Limits exist from the right and the left, but the function is not continuous from either side.

1.5.2 Function Extensions

Recall we like continuous functions because we can use their values at one point to approximate the values they should have at nearby points. And we observed that this is really unhelpful at any point where the function isn't defined. So if we have a function that's continuous everywhere it's defined, we'd like to replace it with a function that is continuous—and defined—everywhere.

Definition 1.72. We say that g is an *extension* of f if the domain of g contains the domain of f, and $g(x) = f(x)$ whenever $f(x)$ is defined.

In general, we can only extend a function to be continuous at all real numbers if the only discontinuities were removable. This is why we call discontinuities like that "removable".

Example 1.73. Let $f(x) = \frac{x^2-1}{x-1}$. Can we define a function g that agrees with f on its domain, and is continuous at all reals?

f is continuous everywhere on its domain, and is undefined at $x = 1$. We can see that $g(x) = x + 1$ will give the same value as f everywhere on f's domain, and it is continuous since it is a polynomial. Thus g is a continuous extension of f to all reals.

Alternatively, we could compute that $\lim_{x\to 1} f(x) = 2$. Then we define

$$h(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1. \end{cases}$$

The function $h(x)$ is defined at all reals, and since it is continuous at 1 by our computation, it is continuous everywhere. It also must extend f since it is just defined to be f everywhere in the domain of f. So h is a continuous extension of f to all reals.

Importantly, g and h are actually the same function, since they give the same output for every input. There is at most one continuous extension of any given function; but there are multiple ways to describe that extension.

Example 1.74. The function $f(x) = 1/x$ is continuous on its domain, but we cannot extend it to a function continuous at all reals, because the limit at 0 does not exist.

Example 1.75. Let $f(x) = \frac{x^2-4x+3}{x-3}$. Can we extend f to a function continuous at all reals?

Answer: f is continuous at all reals except $x = 3$. But the function $g(x) = x - 1$ is the same everywhere except for 3, and is continuous at 3.

Example 1.76. Let

$$g(x) = \begin{cases} x^2 + 1 & x > 2 \\ 9 - 2x & x < 2 \end{cases}$$

Can we extend this to a continuous function on all reals?

Answer: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 9 - 2x = 5$, and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 + 1 = 5$, so the limit at 2 exists. Thus we can extend g to

$$g_f(x) = \begin{cases} x^2 + 1 & x \geq 2 \\ 9 - 2x & x \leq 2 \end{cases}$$

which is continuous at all reals.

1.6 Trigonometry and the Squeeze Theorem

We now want to look at limits of trigonometric functions. Fortunately, they behave *mostly* how we want them to.

Proposition 1.77. *If a is a real number, then* $\lim_{x\to a} \sin(x) = \sin(a)$ *and* $\lim_{x\to a} \cos(x) = \cos(a)$.

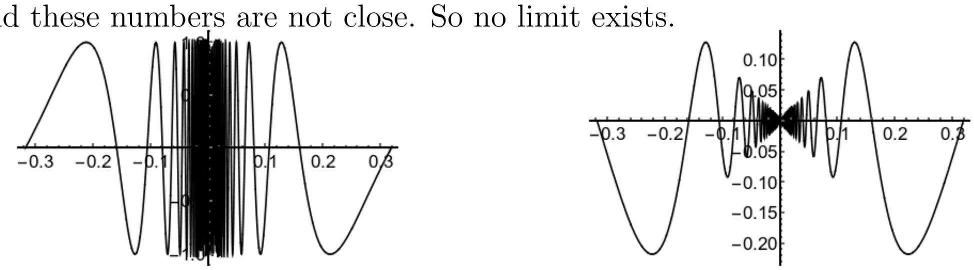
In fact, since trigonometric functions are just ways of combining sine and cosine, essentially all trigonometric functions behave this way where they are defined.

Example 1.78. $\lim_{x\to\pi} \cos(x) = -1$.
$\lim_{x\to\pi} \tan(x) = 0$.

But where the functions are not defined, sometimes very odd things can happen. We've seen a graph of $\sin(1/x)$ before, in section **??**. We said that the function wasn't continuous at 0. In fact, no limit exists there.

Suppose a limit does exist at zero; specifically, let's suppose that $\lim_{x\to 0} \sin(1/x) = L$. Then if x is close to 0, it must be the case that $\sin(1/x)$ is close to L.

But however close we want x to be to 0, we can find a $x_1 = \frac{1}{(2n+1/2)\pi}$, and then $\sin(1/x_1) = \sin((2n + 1/2)\pi) = \sin(\pi/2) = 1$. But we can also find an $x_2 = \frac{1}{(2n+3/2)\pi}$ so that $\sin(1/x_2) = \sin(2n\pi + 3\pi/2) = \sin(3\pi/2) = -1$. So L must be really close to 1 and really close to -1, and these numbers are not close. So no limit exists.



Left: graph of $\sin(1/x)$, Right: graph of $x\sin(1/x)$

In contrast, from the graph it appears that $\lim_{x\to 0} x\sin(1/x)$ does exist. We can't possibly prove this by replacing $x\sin(1/x)$ with an almost identical function and plugging values in:

the function is gross and complicated, and any almost identical function will also be gross and complicated.

But we can easily see that $\lim_{x\to 0} x = 0$. This doesn't mean that $\lim_{x\to 0} xf(x) = 0$ for any $f(x)$; if $f(x)$ gets really big then it can "cancel out" the x term getting very small. (A good example of this is $\lim_{x\to 0} x\frac{1}{x}$, which is of course 1).

But if we can prove that the second term, which in this case is $\sin(1/x)$, does *not* get really big, then the entire limit will have to go to zero. We make this intuition precise with the following important theorem:

Theorem 1.79 (Squeeze Theorem). *If $f(x) \leq g(x) \leq h(x)$ near a (except possibly at a), and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

To use the Squeeze Theorem, we need to do two things:

1. Find a lower bound and an upper bound for the function we're interested in; and
2. show that their limits are equal.

We usually do this by factoring the function we care about into two pieces, where one goes to zero and the other is bounded, and thus doesn't get infinitely big.

In this case, we know that $-1 \leq \sin(1/x) \leq 1$ by properties of $\sin(x)$. We "want" to multiply both sides of the equation by x to get $-x \leq x\sin(1/x) \leq x$, but this is actually incorrect when x is negative. In general, it's hard to reason about inequalities when negative numbers are involved, so we use absolute values to make sure we don't have to worry about it:

$$-|x| \leq x\sin(1/x) \leq |x|$$

Then we can compute that $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$ and so by the squeeze theorem, $\lim_{x\to 0} x\sin(1/x) = 0$.

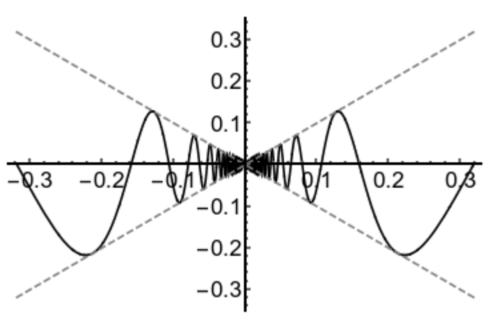


Figure 1.14: A graph of $x\sin(1/x)$ with $|x|$ and $-|x|$

This means that we can extend the function $x\sin(1/x)$ to be continuous at all reals, by defining

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Remark 1.80. There is an argument people make sometimes that looks like the squeeze theorem, but is actually wrong. People reason:

$$\begin{aligned} -|x| &\leq x\sin(1/x) \leq |x| \\ \lim_{x\to 0} -|x| &\leq \lim_{x\to 0} x\sin(1/x) \leq \lim_{x\to 0} |x| \\ 0 &\leq \lim_{x\to 0} x\sin(1/x) \leq 0 \end{aligned}$$

and conclude that $\lim_{x\to 0} x\sin(1/x) = 0$.

However, this reasoning only works if you already know the limit exists. Compare:

$$\begin{aligned} -1 &\leq \sin(1/x) \leq 1 \\ \lim_{x\to 0} -1 &\leq \lim_{x\to 0} \sin(1/x) \leq \lim_{x\to 0} 1 \\ -1 &\leq \lim_{x\to 0} \sin(1/x) \leq 1. \end{aligned}$$

This uses the same reasoning, but the third statement doesn't actually make any sense because the limit doesn't exist. (Imagine writing that $-1 \leq$ green ≤ 1, for instance).

Example 1.81. Using the Squeeze Theorem, show that $\lim_{x\to 3}(x-3)\frac{x^2}{x^2+1} = 0$.

We could in fact do this without the squeeze theorem, but we also can use squeeze.

We divide the function into two parts. We see that $(x-3)$ approaches zero, so we need to bound the other factor.

We know that $0 \leq x^2 \leq x^2+1$ and so $0 \leq \frac{x^2}{x^2+1} \leq 1$ for any x. We want to multiply through by $x-3$, but that only works if $x > 3$. So we use absolute values to keep everything correct and get

$$0 \leq \left|(x-3)\frac{x^2}{x^2+1}\right| \leq |x-3|.$$

Then $\lim_{x\to 3} 0 = \lim_{x\to 3} -|x-3| = 0$, and so by the squeeze theorem $\lim_{x\to 3}(x-3)\frac{x^2}{x^2+1} = 0$.

Example 1.82. What is

$$\lim_{x\to 1} \frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}?$$

The top goes to zero and the bottom is bounded, so this looks like a squeeze theorem problem. If you have trouble seeing this, it may help to rewrite the problem as

$$\lim_{x\to 1}(x-1)\frac{1}{2+\sin\left(\frac{1}{x-1}\right)}.$$

We know that $-1 \leq \sin\left(\frac{1}{x-1}\right) \leq 1$ and so $1 \leq 2+\sin\left(\frac{1}{x-1}\right) \leq 3$, and thus

$$\begin{aligned}
1 &\geq \frac{1}{2+\sin\left(\frac{1}{x-1}\right)} \geq \frac{1}{3} \\
|x-1| &\geq \frac{|x-1|}{2+\sin\left(\frac{1}{x-1}\right)} \geq \frac{|x-1|}{3} \\
|x-1| &\geq \left|\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}\right| \geq \frac{|x-1|}{3}
\end{aligned}$$

since the denominator is always positive. But $\lim_{x\to 1}|x-1| = \lim_{x\to 1}\frac{|x-1|}{3} = 0$, so by the squeeze theorem

$$\lim_{x\to 1}\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)} = 0.$$

Example 1.83. Prove that $\lim_{x\to 3}(x-3)\left(5\sin\left(\frac{1}{x-3}\right)-2\right) = 0$.

We know that

$$\begin{aligned}
-1 &\leq \sin\left(\frac{1}{x-3}\right) \leq 1 \\
-5 &\leq 5\sin\left(\frac{1}{x-3}\right) \leq 5 \\
-7 &\leq 5\sin\left(\frac{1}{x-3}\right) - 2 \leq 3.
\end{aligned}$$

We want to multiply through by $x-3$, but this causes problems when $x < 3$ and thus $x-3 < 0$. So first we put absolute values on everything.

But there's a subtlety here. We know our bad term is between -7 and 3. But when we take absolute values, that doesn't make it larger than $|-7|$ and smaller than $|3|$—no numbers satisfy those rules. Instead, we know that since we've added absolute values, everything will be bigger than zero. This gives us a lower bound.

For the upper bound, we care about how far away from zero we can get. One way to see this is that if $5\sin\left(\frac{1}{x-3}\right)-2 > 0$, we know that it must be less than 3; but if $5\sin\left(\frac{1}{x-3}\right)-2 < 0$, we know it must be bigger than -7, so the absolute value is < 7. So overall we get the bounds

$$0 \leq \left|(x-3)\left(5\sin\left(\frac{1}{x-3}\right)-2\right)\right| \leq |7(x-3)|.$$

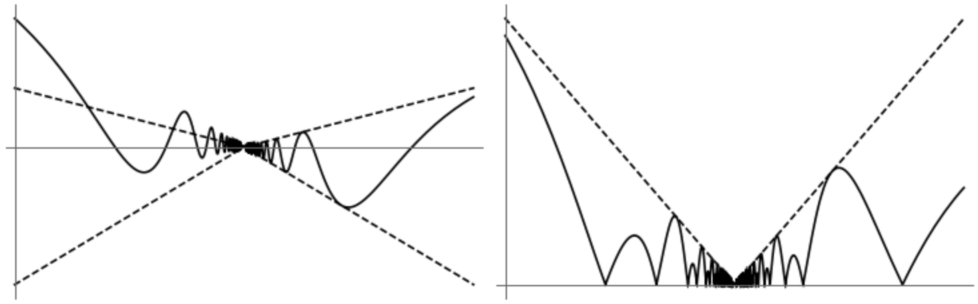


Figure 1.15: Left: $-7|x-3|$ is a fine lower bound, but $3|x-3|$ isn't an upper bound. Right: After we take absolute values, we see that $7|x-3|$ has the smallest coefficient we could possibly use and still get an upper bound.

Now we can compute that $\lim_{x\to 3} 0 = 0$ and $\lim_{x\to 3} |7(x-3)| = 0$, so by the squeeze theorem we know that $\lim_{x\to 3}(x-3)\left(5\sin\left(\frac{1}{x-3}\right)\right) = 0$.

Example 1.84. What is $\displaystyle\lim_{x\to -1}(x+1)\cos\left(\frac{x^5-3x^2+e^x-1700+(2+x)^{(1+x)^x}}{(x+1)^{27.2}}\right)$?

This looks complicated but is actually quite simple. $-1 \leq \cos(y) \leq 1$ for any y, including $y = x^5 - 3x^2 + e^x - 1700 + x^{x^x}$. Thus we have

$$\begin{aligned}0 &\leq |\cos(y)| \leq 1\\ 0 &\leq |(x+1)\cos(y)| \leq |x+1|.\end{aligned}$$

Then we know that $\lim_{x\to -1} 0 = \lim_{x\to -1}|x+1| = 0$. Thus by the squeeze theorem,

$$\lim_{x\to -1}|(x+1)\cos(x^5-3x^2+e^x-1700+x^{x^x})| = 0,$$

and thus

$$\lim_{x\to -1}(x+1)\cos(x^5-3x^2+e^x-1700+x^{x^x}) = 0.$$

Example 1.85. What is

$$\lim_{x\to 0}\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)}?$$

This is a trick question. Here we have no concerns about zeroes in the denominator or points outside of the domain, we can repeatedly apply limit laws:

$$\begin{aligned}\lim_{x\to 0}\frac{x-1}{2+\sin\left(\frac{1}{x-1}\right)} &= \frac{\lim_{x\to 0}(x-1)}{\lim_{x\to 0} 2+\sin\left(\frac{1}{x-1}\right)}\\ &= \frac{-1}{2+\sin\left(\lim_{x\to 0}\frac{1}{x-1}\right)}\\ &= \frac{-1}{2+\sin(-1)} = \frac{-1}{2-\sin(1)}.\end{aligned}$$

Remark 1.86. Notice that we don't conclude that since $f(x) \leq g(x) \leq h(x)$ then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x)$. This is in fact not always true; it's only true if the middle limit exists, which is what we're trying to prove! So we just compute the outer two limits, and then invoke the squeeze theorem.

Example 1.87. $\lim_{x\to+\infty} \frac{\sin(x)}{x}$ exists, by the squeeze theorem.

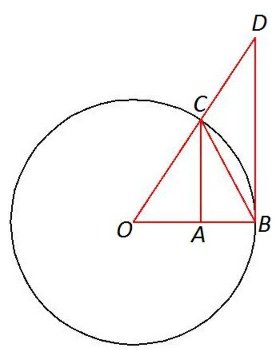
For large x we have $\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$, and $\lim_{x\to+\infty} \frac{-1}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$. So by the squeeze theorem $\lim_{x\to+\infty} \frac{\sin(x)}{x} = 0$.

You might notice this is *exactly the same proof* we gave for $\lim_{x\to 0} x \sin(1/x)$. This is not a coincidence, since the two functions are the same after the substitution $y = 1/x$.

There is one more important limit involving sin:

Proposition 1.88 (Small Angle Approximation)**.**

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$



Proof. We'll assume x is small and positive; this all still works if x is small and negative, with different signs. Our diagram is of a circle with radius 1.

Let x be the measure of angle AOC in our diagram. Observe that $\sin x$ is precisely the length of the line segment AC by definition, and so triangle BOC has area $\sin x/2$. The area of the entire circle is π and so the area of the wedge from B to C is $\pi x/2\pi = x/2$. Since the triangle is contained in the wedge, we have $\sin x/2 \leq x/2$ and thus $\sin x/x \leq 1$.

Note that AC is $\sin x$ and AO is $\cos x$, so AC over AO is $\sin(x)/\cos(x) = \tan(x)$. By similarity, we have DB $= \tan x$, and the area of triangle BOD is $\tan x/2$. Since the wedge from B to C is contained in this triangle, we have $x/2 \leq \tan x/2$ and thus $\cos x \leq \sin x/x$.

Thus $\cos x \leq \frac{\sin x}{x} \leq 1$. But $\lim_{x\to 0} \cos x = 1$, so by the squeeze theorem we have

$$1 \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1$$

and thus get the desired result. □

Remark 1.89. This means that the function

$$f(x) = \begin{cases} \sin(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is a continuous extension of $\sin(x)/x$ to all reals.

Example 1.90. $\lim_{x\to 0} \frac{\sin(2x)}{2x} = 1$.

Example 1.91. What is $\lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x}$?

We can write

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(4x)\sin(6x)}{\sin(2x)x} &= \lim_{x\to 0} \frac{\sin(4x)/4x \cdot \sin(6x)/6x \cdot 24x^2}{\sin(2x)/2x \cdot 2x \cdot x} \\ &= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{\sin 6x}{6x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{24x^2}{2x^2} \\ &= 1 \cdot 1 \cdot 1 \cdot 12 = 12. \end{aligned}$$

Here we are simply pairing off the $\sin(y)$'s with ys and then collecting the remainder into the last term.

Example 1.92. What is $\lim_{x\to 0} \frac{x}{\cos(x)}$?

This problem is actually easy. We can just plug in 0 for x and get $\lim_{x\to 0} \frac{x}{\cos(x)} = \frac{0}{1} = 0$.

In contrast, $\lim_{x\to 0} \frac{\cos(x)}{x}$ is mildly tricky, and we're not ready to do it yet. We'll discuss this sort of limit in section **??**.

Example 1.93. What is $\lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)}$?

When we see a tangent in a problem, it is often helpful to rewrite it in terms of sin and cos. We can then collect terms:

$$\begin{aligned} \lim_{x\to 0} \frac{x\sin(2x)}{\tan(3x)} &= \lim_{x\to 0} \frac{x\sin(2x)}{\sin(3x)/\cos(3x)} \\ &= \lim_{x\to 0} \frac{3x}{\sin 3x} \cdot \frac{\sin(2x)\cos(3x)}{3} = 1 \cdot \frac{0}{3} = 0. \end{aligned}$$

Example 1.94. What is $\lim_{x\to 3} \frac{\sin(x-3)}{x-3}$?

This is a small angle approximation again, since $x - 3$ is approaching zero. Thus the limit is 1.

Example 1.95. What is $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$?

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned} \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} &= \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9} \\ &= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6. \end{aligned}$$

Example 1.96. What is $\lim_{x\to 0} \frac{1-\cos x}{x}$?

We can see that the limits of the top and the bottom are both 0, so this is an indeterminate form. We can't use the small angle approximation directly because there is no sin here at all. But we can fix that by multiplying by the conjugate.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))}$$
$$= \lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)} = \frac{0}{2} = 0.$$

1.7 Infinite Limits

A few times in the past couple sections we've talked about vertical asymptotes, or functions going to infinity. In this section we want to look at exactly what that means. Some limits deal with infinity as an output, and others deal with it as an input (or both).

Remark 1.97. Recall that infinity is not a number. Sometimes while dealing with infinite limits we might make statements that appear to treat infinity as a number. But it's not safe to treat ∞ like a true number and we will be careful of this fact.

1.7.1 Limits To Infinity

Definition 1.98. We write

$$\lim_{x\to a} f(x) = +\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily large (and positive).

We write

$$\lim_{x\to a} f(x) = -\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily negative.

We write

$$\lim_{x\to a} f(x) = \pm\infty$$

to indicate that as x gets close to a, the values of $f(x)$ get arbitrarily positive or negative. We usually use this when both occur.

Remark 1.99. Important note: If the limit of a function is infinity, the limit *does not exist.* This is utterly terrible English but I didn't make it up so I can't fix it. All the theorems that say "If a limit exists" are not including cases where the limit is infinite.

Lemma 1.100. *Let $f(x), g(x)$ be defined near a, such that $\lim_{x\to a} f(x) = c \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then*

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \pm\infty.$$

Further, assuming $c > 0$ then the limit is $+\infty$ if and only if $g(x) \geq 0$ near a, and the limit is $-\infty$ if and only if $g(x) \leq 0$ near a. If $c < 0$ then the opposite is true.

Remark 1.101. If the limit of the numerator is zero, then this lemma is *not useful*. That is one of the "indeterminate forms" which requires more analysis before we can compute the limit completely.

Example 1.102. What is $\lim_{x\to 3} \frac{-1}{\sqrt{x-3}}$? We see the top goes to 1 and the bottom goes to 0, so the limit is $\pm\infty$. Since the denominator is always positive and the numerator is negative, the limit is $-\infty$.

We have to be careful while working these problems: the limit laws that work for finite limits don't always work here, since the limit laws assume that the limits exist, and these do not. In particular, adding and subtracting infinity *does not work*. Instead, we need to arrange the function into a form where we can use lemma **??**.

Example 1.103. We already know that $\lim_{x\to 0} 1/x = \pm\infty$.

1. If we take $\lim_{x\to 0} 1/x - 1/x$, we could say the limit is $\pm\infty - \pm\infty$, but this is silly—the limit is actually 0.

2. In contrast, $\lim_{x\to 0} 1/x + 1/x = \lim_{x\to 0} 2/x = \pm\infty$. We don't add the infinities together.

3. And $\lim_{x\to 0} 1/x + 1/x^2$ is the trickiest. We have a $\pm\infty$ plus a $+\infty$. But again we can't add infinities—we need to combine them into one term.

   $$\lim_{x\to 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x\to 0} \frac{x+1}{x^2} = +\infty$$

   since the numerator approaches 1 and the denominator approaches 0, but is always positive.

   We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ "faster" than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

We organize our thinking about these situations in terms of the "indeterminate forms", which are: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty \pm \infty, 1^\infty, \infty^0$. Notice that none of these are actual numbers, and they can never be the correct answer to pretty much any question.

More importantly, indeterminate forms don't even tell us what the answer should be; if plugging in gives you one of those forms, the true limit could potentially be pretty much anything. We have to do more work to get our functional expression into a determinate form. As a general rule, we use algebraic manipulations to get a form of $\frac{0}{0}$, then factor out and cancel $(x-a)$ until either the numerator or the denominator is no longer 0.

Remark 1.104. Neither $\frac{0}{1}$ nor $\frac{1}{0}$ is an indeterminate form. $\frac{0}{1}$ is just a number, equal to 0. $\frac{1}{0}$ is not a number and is never the correct answer to a question, but it's also not indeterminate. By lemma **??**, if $\lim f(x) = 1$ and $\lim g(x) = 0$ then $\lim f(x)/g(x) = \pm\infty$.

Similarly, $\frac{0}{\infty}$ and $\frac{\infty}{0}$ are also not numbers but not indeterminate. The first suggests the limit is 0; the second suggests the limit is $\pm\infty$.

The form $\infty \cdot \infty$ mostly works fine, and gives you another ∞ whose sign depends on the signs of the ∞s you're multiplying. But again, $\infty \cdot \infty$ is never the actual answer to any actual question.

Example 1.105. What is $\lim_{x\to -2} \frac{1}{x+2} + \frac{2}{x(x+2)}$? This looks like $\infty + \infty$ so we have to be careful. We have

$$\begin{aligned} \lim_{x\to -2} \frac{1}{x+2} + \frac{2}{x(x+2)} &= \lim_{x\to -2} \frac{x}{x+2} + \frac{2}{x(x+2)} \\ &= \lim_{x\to -2} \frac{x+2}{x(x+2)} = \lim_{x\to -2} \frac{1}{x} = \frac{-1}{2}. \end{aligned}$$

Example 1.106. $\lim_{x\to 3^+} \frac{1}{(x-3)^3} = +\infty$: the limit of the top is 1, and the limit of the bottom is 0, so the limit is $\pm\infty$. But when $x > 3$ the denominator is ≥ 0, so the limit is in fact $+\infty$. Conversely $\lim_{x\to 3^-} \frac{1}{(x-3)^3} = -\infty$ since when $x < 3$ we have $(x-3)^3 \leq 0$.

$\lim_{x\to -1^+} \frac{1}{(x+1)^4} = +\infty$. And $\lim_{x\to -1^-} \frac{1}{(x+1)^4} = +\infty$. Thus $\lim_{x\to -1} \frac{1}{(x+1)^4} = +\infty$.

1.7.2 Limits at infinity

A related concept is the idea of limits "at" infinity, which answers the question "what happens to $f(x)$ when x gets very big?" We can formally define this in terms of ϵ.

Definition 1.107. Let f be a function defined for (a, ∞) for some number a. We write

$$\lim_{x\to +\infty} f(x) = L$$

to indicate that when x is large enough, the values of $f(x)$ get arbitrarily close to L. Formally, if for every $\epsilon > 0$ there is a $M > 0$ so that if $x > M$ then $|f(x) - L| < \epsilon$.

We can write similar definitions for $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\pm\infty} f(x)$, and talk about when these limits are themselves $\pm\infty$. But here we'll skip over the formal definition and simply think informally.

In principle, we want to do the same thing we did for finite limits. But instead of having zeros on the top and bottom of a fraction, we often have infinities as well. So we want to "cancel" an infinity from the top and the bottom of the fraction. We usually do this by dividing the top and bottom by x. Then we can use the following crucial fact:

Fact 1.108. $\lim_{x\to\pm\infty} \frac{1}{x} = 0$.

This combined with tools we already have is enough to do pretty much any calculuation here.

Example 1.109. If we want to calculuate $\lim_{x\to+\infty} \frac{1}{\sqrt{x}}$, we see that

$$\lim_{x\to+\infty} \frac{1}{\sqrt{x}} = \sqrt{\lim_{x\to+\infty} \frac{1}{x}} = \sqrt{0} = 0.$$

Example 1.110. What is $\lim_{x\to+\infty} \frac{x}{x^2+1}$?

This problem illustrates the primary technique we'll use to solve infinite limits problems. It's difficult to deal with problems that have variables in the numerator and denominator, so we want to get rid of at least one. Thus we will divide out by xs on the top and the bottom until one has none left:

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{x/x}{x^2/x + 1/x} = \lim_{x\to+\infty} \frac{1}{x + \frac{1}{x}} = \lim_{x\to+\infty} \frac{1}{x} = 0.$$

Example 1.111. Some more examples of this technique:

$$\lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1} = 1.$$
$$\lim_{x\to-\infty} \frac{x}{3x+1} = \lim_{x\to-\infty} \frac{1}{3+\frac{1}{x}} = \frac{1}{3}.$$

Example 1.112. What is $\lim_{x\to+\infty} \frac{x^{3/2}}{\sqrt{9x^3+1}}$? This one is a bit tricky. We want to divide the top and bottom by $x^{3/2}$. Then we can pull the factor *inside* the square root sign.

$$\lim_{x\to+\infty} \frac{x^{3/2}}{\sqrt{9x^3+1}} = \lim_{x\to+\infty} \frac{1}{\sqrt{9+1/x^{3/2}}} = \frac{1}{\sqrt{9+0}} = \frac{1}{3}.$$

Example 1.113. Sometimes it's a bit harder to see how this works. For instance, what is $\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}}$? It's not obvious, but we use the same technique:

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to+\infty} \frac{1}{\sqrt{x^2/x^2+1/x^2}} = \lim_{x\to+\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = 1.$$

Example 1.114. What is $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$?

We can do the same thing, but we have to be *very careful.* Remember that if $x < 0$ then $\sqrt{x^2} \neq x$! Instead, $x = -\sqrt{x^2}$. Thus we have

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}/(-\sqrt{x^2})} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}} = -1.$$

When we encounter new functions, one of the ways we will often want to characterize them is by computing their limits at $\pm\infty$. Sometimes these limits do not exist.

Example 1.115. $\lim_{x\to+\infty} \sin(x)$ does not exist, since the function oscillates rather than settling down to one limit value.

$\lim_{x\to+\infty} x\sin(x)$ also does not exist; this function oscillates more and more wildly as x increases.

But $\lim_{x\to+\infty} \frac{1}{x}\sin(x)$ does in fact exist. We can prove this with the squeeze theorem: we can see that $\frac{-1}{x} \leq \frac{1}{x}\sin(x) \leq \frac{1}{x}$, and we know that $\lim_{x\to+\infty} \frac{-1}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$. So by the Squeeze Theorem, $\lim_{x\to+\infty} \frac{1}{x}\sin(x) = 0$.

Another technique that will also often appear in these limits is combining a sum or difference into one fraction. If we have a sum of two terms that both have infinite limits, we need to combine or factor them into one term to see what is happening.

Example 1.116. What is $\lim_{x\to-\infty} x - x^3$?

Each term goes to $-\infty$, so this is a difference of infinities and thus indeterminate. But we can factor: $\lim_{x\to-\infty} x(1-x^2)$. The first term goes to $-\infty$ and the second term also goes to $-\infty$, so we expect that their product will go to $+\infty$. Thus $\lim_{x\to-\infty} x - x^3 = +\infty$.

To be precise, I should compute:

$$\lim_{x\to-\infty} x - x^3 = \lim_{x\to-\infty} \frac{x-x^3}{1} = \lim_{x\to-\infty} \frac{1/x^2-1}{1/x^3}.$$

We see the limit of the top is -1 and the limit of the bottom is 0, so the limit of the whole is $\pm\infty$. In fact the bottom will always be negative (since $x \to -\infty$), and thus the limit is $+\infty$.

Example 1.117. What is $\lim_{x\to+\infty} \sqrt{x^2+1} - x$?

We might want to try to use limit laws here, but we would get $+\infty - +\infty$ which is not defined (and is one of the classic indeterminate forms). Instead we need to combine our expressions into one big fraction.

$$\begin{aligned}\lim_{x\to+\infty} \sqrt{x^2+1} - x &= \lim_{x\to+\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to+\infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1}+x}.\end{aligned}$$

The numerator is 1 and the denominator approaches $+\infty$ so the limit is 0. This tells us that as x increases, x and $\sqrt{x^2+1}$ get as close together as we wish.

You may have noticed the appearance of our old friend, multiplication by the conjugate. We will often use that technique in this sort of problem.

Example 1.118. What is $\lim_{x\to+\infty} \sqrt{x^2+x+1} - x$?

$$\begin{aligned}\lim_{x\to+\infty} \sqrt{x^2+x+1} - x &= \lim_{x\to+\infty} \left(\sqrt{x^2+x+1} - x\right) \frac{\sqrt{x^2+x+1}+x}{\sqrt{x^2+x+1}+x} \\ &= \lim_{x\to+\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x} = \lim_{x\to+\infty} \frac{x+1}{\sqrt{x^2+x+1}+x} \\ &= \lim_{x\to+\infty} \frac{1+1/x}{\sqrt{1+1/x+1/x^2}+1} = \frac{1}{2}.\end{aligned}$$