6 Applications of Integrals

Now that we know how to compute integrals, we want to finish the course by talking about why integrals are so useful. We'll start with the idea of computing area, which was the original motivation for integration; but we'll see they also answer many other questions.

6.1 Finding Areas

Recall that we originally constructe the integral to find the area of some shape, in particular of shapes that lie under the graph of some function. We can use the same tools to find the area of a region that is not, properly speaking, the graph of one function.

The simplest (well, second-simplest) case is the case where we want the area of a region that lies in between the graph of two functions. We can approximate area by drawing, as before, a great many skinny rectangles which are approximately the right height to cover our region. If our region lies in between two functions f and g, the combined area of our rectangles is

$$\sum_{i=1}^{n}(x_i - x_{i-1})(f(x_i^*) - g(x_i^*))$$

and as the number of rectangles increases this approximation gets increasingly good. We say the area of the region is

$$A = \lim_{n\to+\infty}\sum_{i=1}^{n}(x_i - x_{i-1})(f(x_i^*) - g(x_i^*)).$$

You may recognize this formula as the integral of the function $f - g$; indeed, if we have a region with x coordinates varying from a to b and y coordinates varying from $g(x)$ to $f(x)$, then its area is $\int_a^b (f(x) - g(x))\,x$.

Remark 6.1. Remember that actual areas are always positive! The integral by itself computes the "signed area"; if you want an actual area you must be careful to make sure you're integrating the correct function.

Example 6.2. Let's start with a trivial example: what's the area of a rectangle with base 3 and height 4? Well, this is $\int_0^4 3\,dx = 3x|_0^4 = 12$, as it should be.

Example 6.3. What is the area of the region between $y = x^3$ and $y = 1/x^2$ between $x = 2$ and $x = 4$?

We have

$$\int_2^4 x^3 - (1/x^2)\,dx = \left(\frac{x^4}{4} + \frac{1}{x}\right)\Bigg|_2^4 = (64 + 1/4) - (4 + 1/2) = 60 - 1/4 = 239/4.$$

Sometimes (usually!) we need to have a visual idea of what our region looks like before we can set up an appropriate integral.

Example 6.4. What is the area of the region bounded by $y = x$ an $y = x^2$?

After we draw a picture, we see that these two graphs enclose a region between $x = 0$ and $x = 1$, and that in that region, $x \geq x^2$. So we compute the integral

$$\int_0^1 x - x^2\, dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Example 6.5. Compute the total area of the "valley" between two peaks of the sine function.

We see that this area is the area of the region between $y = 1$ and $y = \sin x$ between $\pi/2$ and $5\pi/2$. (There are other ways to set this up, but this way works). So we compute

$$\int_{\pi/2}^{5\pi/2} 1 - \sin x\, dx = x + \cos(x)|_{\pi/2}^{5\pi/2} = (5\pi/2 + 0) - (\pi/2 + 0) = 2\pi.$$

Sometimes you have to break your region up into separate pieces/integrals

Example 6.6. What is the area of the region bounded by $y = x^2, y = 2 - x$, and $y = 0$?

We sketch the region and see that we get a sort of collapsed triangle. We compute

$$\begin{aligned} A &= \int_0^1 x^2\, dx + \int_1^2 (2 - x)\, dx = \frac{x^3}{3}|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 \\ &= \frac{1}{3} - 0 + (4 - 2) - (2 - 1/2) = \frac{5}{6}. \end{aligned}$$

We can also do the same problem another way. Notice that we might as well write $x = \sqrt{y}, x = 2 - y$. So we can just as well integrate with respect to y—that is, draw our rectangles stretching horizontally instead of vertically. We have

$$A = \int_0^1 (2 - y) - \sqrt{y}\, dy = \left(2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right)\Big|_0^1 = \left(2 - \frac{1}{2} - \frac{2}{3}\right) - 0 = \frac{5}{6}.$$

As expected, we get the same answer.

Remark 6.7. In general, if you have straight line or point boundaries on opposite sides, you should integrate between them. In general, if you can write something as the difference of two functions one way and not the other way, you should do that.

Example 6.8. What is the area of the region between $y^2 = x + 3$ and $y = x - 3$?

These curves intercept when $y^2 = y + 6$, which happens when $y = 3$ or $y = -2$, and thus at $(6, 3)$ and $(1, -2)$. It's more natural to integrate with respect to y, so we write

$$A = \int_{-2}^{3} (y+3) - (y^2 - 3)\, dy = \int_{-2}^{3} 6 + y - y^2\, dy$$
$$= \left(6y + \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{-2}^{3} = \left(18 + \frac{9}{2} - 9\right) - \left(-12 + 2 + \frac{8}{3}\right) = \frac{27}{2} + 10 - \frac{8}{3} = \frac{125}{6}$$

Example 6.9. What is the area of the region bounded by $y = x^2 + 1, y = 17 - x^2$, and $y = 1$?

We first draw the region, and see a sort of sideways triangle with a base at $x = 1$ and a point at $(\sqrt{8}, 9)$, with x varying from 1 to 17. We have two options: integrate with respect to x, or with respect to y by writing $x = \sqrt{y-1}$ and $x = \sqrt{17 - y}$. The second involves breaking our region into two integrals, and gives us

$$A = \int_2^9 \sqrt{17 - y} - 1\, dy + \int_9^{16} \sqrt{17 - y} - 1\, dy,$$

which is doable but pretty ugly.

Instead, if we integrate with respect to x, we get

$$A = \int_1^{\sqrt{8}} (17 - x^2) - (x^2 - 1)\, dx = \int_1^{\sqrt{8}} 18 - 2x^2\, dx$$
$$= 18x - \frac{2}{3}x^3\Big|_1^{\sqrt{8}} = 36\sqrt{2} - 32\sqrt{2}/3 - 18 + 2/3 = \frac{76\sqrt{2} - 52}{3}.$$

6.1.1 Economic Application: Consumer and Producer Surplus

This sort of area computation often comes up in the context of computing consumer and producer surpluses. Economists will often draw supply and demand curves to describe a market. The market clears when the price

Example 6.10. Suppose the demand for a product is given by $p = d(q) = 20 - .05q$ and the supply for the same product is given by $p = s(q) = 2 + .0002q^2$. For both functions, q is the quantity and p is the price, in dollars.

Then demand when $d(q) = s(q)$, so $20 - .05q = 2 + .0002q^2$ and thus we have $.0002q^2 + .05q - 18 = 0$. Using the quadratic formula, we get the solutions -450 and 200, so the equilibrium happens when 200 units are produced.

What then is the surplus? The total surplus is

$$\begin{aligned}\int_0^{200} d(q) - s(q)\,dq &= \int_0^{200} 18 - .05q - .0002q^2\,dq \\ &= 18q - .025q^2 - \frac{.0002}{3}q^3\Big|0^{200} \\ &= 3600 - 1000 - 533.33 = 2066.67.\end{aligned}$$

Thus the total surplus value created is a bit more than \$2000.

But how is this surplus distributed? We can compute that the equilibrium price is \$10, so we have a consumer surplus of

$$\begin{aligned}\int_0^{200} d(q) - p\,dq &= \int_0^{200} 10 - .05q\,dq \\ &= 10q - .025q^2\Big|_0^{200} \\ &= 2000 - 1000 = 1000.\end{aligned}$$

Conversely we can compute the producer surplus of

$$\begin{aligned}\int_0^{200} p - s(q)\,dq &= \int_0^{200} 8 - .0002q^2\,dq \\ &= 8q - \frac{.0002}{3}q^3\Big|_0^{200} \\ &= 1600 - 533.33 = 1066.67\end{aligned}$$

6.2 The Average Value of a Function

This is a convenient time to address the concept of "average value." If we have some finite collection of numbers, the average is what we get when we add them up, and divide by the number of numbers:

$$\frac{1}{n}\sum_{i=1}^{n} a_i.$$

A function gives us infinitely many numbers; but integration is in some sense a sensible way to add infinitely many numbers up, and so hopefully to average them.

In particular, if we sample the function at n evenly spaced points, our average is

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a}\sum_{i=1}^{n}\frac{b-a}{n}f(x_i^*)$$

which you should recognize as a Riemann sum (times $\frac{1}{b-a}$). If we take the limit—which represents taking the average value after "infinitely many" sample points—we get the following definition:

Definition 6.11. The *average value of a function* f over an interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(t)dt.$$

Example 6.12. What is the average value of $f(x) = x^2$ on $[0, 1]$? We have

$$f_{ave} = \frac{1}{1}\int_0^1 x^2\,dx = \frac{1}{3}.$$

The biggest value is 1, the smallest is 0, and the one in the middle is $\frac{1}{4}$, but the "average" value is $\frac{1}{3}$.

If I have a finite set of numbers and take the average, my average might not be anywhere in the set; for instance, if I roll a six-sided die, the average output will be 3.5, which isn't on the die at all. When I average continuous quantities, however, this can't happen.

Theorem 6.13 (Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there is a number c in $[a, b]$ such that*

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

In other words,

$$\int_a^b f(t)\,dt = f(c)(b-a).$$

Proof. This statement, as well as its name, might look familiar. In fact this is just the mean value theorem from differential calculus repackaged. Let $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a, b]$ and differentiable on (a, b), and so by the Mean Value Theorem there is some c such that $F(b) - F(a) = F'(c)(b-a)$.

But by the Fundamental Theorem of Calculus, $F'(c) = f(c)$. And it's easy to see that $F(b) = \int_a^b f(t)\,dt$, and $F(a) = \int_a^a f(t)\,dt = 0$. So we have

$$\int_a^b f(t)\,dt - 0 = f(c)(b-a).$$

□

Remark 6.14. Geometrically, this essentially tells us that there is some rectangle with the same area as the region under the graph of f. In particular, we can take a rectangle with width $b-a$, whose top edge intersects the graph of our function *somewhere*, and whose area is the same as the area of the region under the curve.

6.3 Applications to Physics

Now we should discuss some physical and other practical processes that are well-described by integration—which is just a fancy way of saying that integrals let us solve these problems.

6.3.1 Multiplication and Units

The fundamental idea here reaches back to the Riemann sum. Recall that the integral computes

$$\int_a^b f(t)\,dt = \lim_{n\to+\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n} = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k.$$

So any time we have a computation that looks like $\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$, we can approach it using an integral.

In particular, the integral generalizes *multiplication*. There are a lot of physical quanties that we can compute as the product of two other things, like distance as the product of speed and time. As long as speed is constant, we can get distance by multiplying speed and time. But if speed is changing, that doesn't quite work.

Instead, we can split our big time interval up into a bunch of little time intervals, and pretend that our speed is constant on each interval. So for a small time interval, we approximate $d_i \approx v(t_i)\Delta t_i$ for the distance traveled in that interval, and we have total distance

$$D \approx \sum_{k=1}^{n} d_k \approx \sum_{k=1}^{n} v(t_i)\Delta t_i.$$

This is of course a Riemann sum, so as we take an integral we get

$$D = \int_a^b v(t)\,dt.$$

Thus we see that total distance traveled is the integral of velocity. (This shouldn't be a surprise, since velocity is the derivative of distance.)

How do the units work out here? Well, velocity is in meters per second, and time is in seconds. So we have d_i measured in (meters per second) times (seconds), which is just meters. In general, if we are computing $I = \int_a^b f(t)\,dt$, then the units of I are the units of $f(t)$ times the units of t—output units times input units. (Compare to the derivative, where the units of $f'(a)$ are output units divided by input units.)

6.3.2 Work

In physics, *force* is the product of mass and acceleration; intuitively, force is what causes a mass to accelerate, and the more acceleration/the more massive the object, the more force is required. This is often written $F = m \cdot a$, but in our context it is better to say that the position of an object is given by the function $s(t)$, and then $F = m \cdot \frac{d^2s}{dt^2}$, since acceleration is the second derivative of position.

Remark 6.15. In the SI system, mass is measured in kilograms, and force is measured in newtons, where $N = kg \cdot m/s^2$. In the Imperial system most Americans use, the pound is a unit of force; the unit of mass is the *slug*, and one pound is one slug-foot per second squared. I bring this up primarily because the name "slug" is funny.

Intuitively, moving things aroudn takes work, and moving them faster takes more work. Formally, we say that *work* is force times distance: the amount of force applied to an object, times the distance the object is moved. The SI unit for work is the Newton-meter or *joule*, which is $J = kg \cdot m^2/s^2$. The imperial unit for work is the foot-pound, which is about 1.36 joules.

If you lift a 2 kg object a meter, then you have to exert $2 \cdot 9.8$ newtons of force (since acceleration due to gravity is $9.8m/s^2$, and thus do 19.6 joules of work. If a 20 pound weight is lifted five feet, than 100 foot-pounds of work are done.

When force is constant, work is easy to calculuate–just multiply the force by the distance. Things become more interesting when the force varies. As usual, we can approximate by chopping the movement up into lots of little pieces, assuming the force is constant on each small piece, and adding them up. That is, if the force at position x is $F(x)$, then when an object moves from a to b the work done is approximately

$$W \approx \sum_{i=1}^{n} F(x_i) \frac{b-a}{n}.$$

This is a Riemann sum, so taking the limit gives an integral: the total work done is

$$\int_a^b F(x)\, dx.$$

Remark 6.16. Unlike most of the geometric integrals we've been doing for the past few weeks, work can be a negative number; this just indicates that the force is in the opposite direction of the motion.

Example 6.17. A particle is controlled by a force field such that the force on it is $x^3 + x$ pounds when it is x feet away from the origin. How much work does it take to move the

particle from $x = 2$ to $x = 4$?

$$W = \int_2^4 x^3 + x\,dx = \frac{x^4}{4} + \frac{x^2}{2}|_2^4 = 64 + 8 - 4 - 2 = 66.$$

Example 6.18. A physical law called Hooke's Law says that the force exerted by a string stretched x units beyond its natural length is kx, where k is the "spring constant" and depends on the particular spring.

Suppose a spring is naturally 20 cm and it takes 50 N to stretch it to 30 cm. How much work is needed to stretch the spring from 30cm to 35cm?

We have $50 = k \cdot .1$ and so $k = 500$. Thus the force when the spring is stretched x meters beyond its normal length is kx, and the work done is

$$W = \int_{.1}^{.15} 500x\,dx = 250x^2|_{.1}^{.15} = 3.125J.$$

Example 6.19. A 50 meter cable has a mass of 50kg and hangs from the top of a cliff. How much work does it take to raise the cable up the cliff?

The thing that makes this difficult is that the mass of the remaining rope depends on how much mass we've lifted already. Conceptually, you can think about having to lift the first meter of rope one meter, and the second meter of rope two meters, etc. Each meter of rope masses 1 kg, so this would give us a Riemann sum

$$W \approx \sum_{i=1}^{50} 1 \cdot 9.8 \cdot i$$

Or more generally

$$W \approx \sum_{i=1}^{n} \Delta x \cdot 9.8 \cdot x_i.$$

Taking the limit gives the integral

$$W = \int_0^{50} 9.8x\,dx = 4.9x^2|_0^{50} = 2500 \cdot 4.9 = 12250J.$$

Example 6.20. A tank of water is shaped like an upside-down pyramid. (No, I don't know why people keep building tanks shaped like upside-down pyramids). The pyramid has a base side length of 4m and a height of 12m, and it is filled with water to a depth of 8m. How much work will it take to pump the water out of the top of the tank? (water has a density of 1000kg / m^3).

Again, to figure out our integral we may want to set up the Riemann sum, or at least fake set it up. Let 0 be the point of the pyramid and 12 be the base (at the top). The

volume of a small cross-sectional volume is $A(h)\Delta h$, thus the mass is $1000A(h)\delta h$ and the force is $1000A(h)\Delta h \cdot 9.8$. The distance we have to pump the water is $12-h$, so the total work on each cross-section is $(12-h)9800A\Delta h$ Newtons.

Now we just have to work out area in terms of height. Using a similar triangles argument, we see that $\frac{s(h)}{h} = \frac{4}{12}$ and thus $s(h) = h/3$, and $A(h) = h^2/9$. We integrate from 0 to 8 becasue we're integrating over the height that contains water. Then we have

$$\int_0^8 (12-h)9800 \cdot h^2/9 \cdot \, dy = \frac{9800}{9}\left(4h^3 - \frac{h^4}{4}\right)|_0^8 = \frac{9800}{9}\left(2048 - 1024 - 0\right) = \frac{10,035,200}{9} J.$$

6.3.3 Hydrostatic Pressure

Another problem we can handle easily with these tools is the idea of water (or fluid) pressure. If you imagine a flat surface submerged in some fluid with density ρ to a depth of d meters, then the weight of the fluid over it is $A\rho dg$ where A is the area of the surface (and thus $Ad\rho$ is the mass of the fluid) and $g = 9.8$ is acceleration due to gravity. We define the pressure to the be the force divided by the area, and thus $P = \frac{F}{A} = \rho dg$.

(In SI units we measure this in Newtons per square meter, otherwise known as Pascals. In Imperial units there are a number of different units used, including "inches of mercury.")

Fact 6.21. *If an object is submerged in a fluid to a given depth, the pressure exerted by the fluid is the same in all directions.*

This means that fluid pressure is effectively a function of height/depth and nothing else. If the pressure is varying and we want to find the total force acting on a surface, we can effectively add up the pressure on each little patch of a surface to find the total force acting on it.

Example 6.22. A 3 by 3 meter square is submerged in water until it is just covered, edge-first. What is the total force the water exerts on the square?

We want to chop the square into strips that are all at roughly the same depth. If we slice the square into three horizontal strips, then the ith strip is roughly at depth i meters and has width 3, and thus has roughly the force $3 \cdot 1 \cdot i \cdot \rho \cdot g$. Adding up the force on all thirty strips gives

$$F \approx \sum_{i=1}^{3} 3 \cdot 1 \cdot i \cdot \rho \cdot g = \sum_{i=1}^{3} 3 \cdot 1000 \cdot 9.8 \cdot h\Delta h$$

In the limit, we get the following integral:

$$\int_0^3 3 \cdot \rho \cdot g \cdot h \, dh = \int_0^3 3000 \cdot 9.8 \cdot h \, dh = 29400(h^2/2)|_0^3 = 29400 \cdot \frac{9}{2} = 132,300.$$

Example 6.23. A cylindrical drum is lying on its side underwater. The drum has radius of 5 feet and is submerged in 20 feet of water. What is the force exerted on one circular face of the drum?

Let's set 0 to be the center of the circle, so that the equation for the circle is $x^2+y^2=25$. Then the width of the object at height y is $2\sqrt{25-y^2}$. The depth at height y is $15-y$ (which ranges from 10 to 20), and the pressure due to water is 62.5 * depth. So we get the integral

$$F = \int_{-5}^{5} 62.5(15-y)2\sqrt{25-y^2}\,dy = 125\int_{-5}^{5} 15\sqrt{25-y^2}\,dy - 125\int_{-5}^{5} y\sqrt{25-y^2}\,dy.$$

The second integral is 0 because $y\sqrt{25-y^2}$ is an odd function. The first integral can be done by setting $y=5\sin\theta$, but we can also observe that it is the integral of a semicircle of radius 5 and thus is equal to 12.5π. So we have

$$F = 125\cdot 15\cdot 12.5\pi = 23437.5lb.$$

6.3.4 Center of Mass

The center of mass of a two dimensional object is, conceptually, the point it can balance on. It is in some sense the "average" location the region occurs.

If the mass of an object occurs in finitely many points, then the center of mass is the weighted average of their locations, where the weighting is by the mass. So if we have particles of mass m_1, m_2, m_3 at points $(x_1,y_1),(x_2,y_2),(x_3,y_3)$, with total mass m, then the x-coordinate of the center of mass of the system is

$$\overline{x} = \frac{1}{m}\sum_{i=1}^{3} m_i x_i = m_1x_1 + m_2x_2 + m_3x_3$$

and the y-coordinate is

$$\overline{y} = \frac{1}{m}\sum_{i=1}^{3} m_i y_i = m_1y_1 + m_2y_2 + m_3y_3$$

As a vocabulary note, we say that each of these m_ix_i or m_iy_i is a *moment* of the mass, and the sum $\sum_{i=1}^{n} m_ix_i$ is the *moment of the system* about the origin in the x-axis.

Example 6.24. We have particles of masses $1,4,5$ at the poitns $(0,0),(3,2),(4,5)$. Then

for the center of mass we have

$$\overline{x} = \frac{1}{10}(1\cdot 0 + 4\cdot 3 + 5\cdot 4) = \frac{32}{10}$$
$$\overline{y} = \frac{1}{10}(1\cdot 0 + 4\cdot 2 + 5\cdot 5) = \frac{33}{10}.$$

We extend this study to calculus. Suppose we have a plate of "uniform density" (i.e. it's all the same material, so bits with the same area wil have the same mass/weight). For concreteness, say the region is given by $a \le x \le b$ and $g(x) \le y \le f(x)$. We'd like to find the center of mass, the point the plate balances perfectly. We can think about how to make it balance in each direction, so we can find the x-coordinate and the y-coordinate separately.

To find the x coordinate of the center of mass, we add up the mass of each vertical strip, weighted by its x-coordinate, just as we did before. The vertical strip has width dx and height $f(x) - g(x)$. Thus each strip has area $(f(x) - g(x))dx$, and we can assume the density is 1 so that it has mass $(f(x) - g(x))dx$ as well. Thus the moments of mass are $x(f(x) - g(x))dx$, and the x-coordinate of the center of mass is

$$\overline{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))dx.$$

To find the y-coordinate, we could do the same thing with respect to y. But if our region is described in terms of a function of x, then this might be awkward. But we can still add up the moment of each *vertical* strip. The strip at x still has area $(f(x) - g(x))dx$, and the "average" position of the strip is the middle of the strip, which is at $\frac{1}{2}(f(x) + g(x))$. So the moment is $\frac{1}{2}(f(x) - g(x))^2\,dx$ and the y-coordinate is

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\,dx.$$

Example 6.25. Find the center of mass of the region bounded by $y = x^2$ and $y = \sqrt{x}$.

The area is

$$A = \int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3}x^{3/2} - \frac{x^3}{3}|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Then we have

$$\overline{x} = 3\int_0^1 x(\sqrt{x} - x^2)\,dx = 3\left(\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right)\Big|_0^1 = 3\left(\frac{2}{5} - \frac{1}{4}\right) = \frac{9}{20}.$$
$$\overline{y} = 3\int_0^1 \frac{1}{2}(\sqrt{x}^2 - (x^2)^2)\,dx = \frac{3}{2}\int_0^1 \left(x - x^4\right)\,dx = \frac{3}{2}\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1$$
$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{9}{20}.$$

Example 6.26. Find the center of mass of the semicircle bounded by $y = \sqrt{r^2 - x^2}$ and $y = 0$ between $x = -r$ and $x = r$.

The area is half the area of a circle, and thus $\frac{1}{2}\pi r^2$. Then we have

$$\begin{aligned}
\overline{x} &= \frac{2}{\pi r^2}\int_{-r}^{r} x\sqrt{r^2-x^2} = 0 \text{ since } x\sqrt{r^2-x^2} \text{ is odd.}\\
\overline{y} &= \frac{2}{\pi r^2}\int_{-r}^{r} \frac{1}{2}(\sqrt{r^2-x^2})^2\,dx = \frac{1}{\pi r^2}\int_{-r}^{r} r^2 - x^2\,dx\\
&= \frac{1}{\pi r^2}\left(r^2x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} = \frac{1}{\pi r^2}\left(r^3 - \frac{r^3}{3} - \left(-r^3 - \frac{r^3}{3}\right)\right) = \frac{1}{\pi r^2}\cdot\frac{4}{3}r^3 = \frac{4r}{3\pi} \approx .42.
\end{aligned}$$

Thus the center of mass is at about $(0, .42)$. The fact that the x coordinate should be 0 is geometrically obvious; the y coordinate is less so.

6.4 Finding Volumes by Cross-Sections

Area is fundamentally length times width, and we computed areas by integrating the length against the width–by which I mean, we wrote the length at a point as a function of the width at that point, and took the integral across the whole width.

Volume is area times height. (Or area times length, depending on your perspective). We will compute volume by finding the area of a cross-section and integrating along the entire length of our shape. Geometrically, the Riemann sum corresponds to slicing our shape into many thin cylinders and adding their areas up.

Remark 6.27. In our terminology, a "cylinder" is any solid that has a flat base and an identical flat top, connected by straight sides at right angles. A traditional circular cylinder qualifies, but so does a rectangular box, and so do stranger shapes.

Definition 6.28. If S is a solid, we say the *cross-sectional area* at a point x is the area of the intersection of our solid with the plane which passes through x and is perpendicular to the x-axis (and thus parallel to the yz plane).

If S is a solid lying between $x = a$ and $x = b$, and $A(x)$ is a function giving the cross-sectional area at x, then we say the *volume* V of S is

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx.$$

Example 6.29. What is the volume of a cone with height 2 and base radius 4?

We draw a picture. By a similar triangles argument, we see that when we are x distance from the point, the radius is $2x$ and thus the area of the cross-section is $4\pi x^2$. Thus the

volume is

$$\int_0^2 4\pi x^2\,dx = \frac{4\pi x^3}{3}|_0^2 = \frac{32}{3}\pi.$$

This matches the formula for the volume of a cone, which is $\frac{1}{3}\pi r^2 h$.

In fact, we can also rederive that formula. If a cone has height h and base radius b, then the radius at x distance from the height is $x\frac{b}{h}$ and the area is $\pi x^2 b^2/h^2$. So the volume of the cylinder is

$$\int_0^h \pi x^2 b^2/h^2\,dx = \pi b^2/h^2 \frac{x^3}{3}|_0^h = \frac{b^2 h\pi}{3}.$$

Example 6.30. What is the volume of a solid with a circular base of radius one, where each cross-section is an equilateral triangle?

Make the circle $x^2 + y^2 = 1$. Then the width of the base of the cross-section at x is $2\sqrt{1-x^2}$. Since $\sin 60° = \sqrt{3}2$, we know the height of each triangle is $\sqrt{3}b/2$, and thus the area of the triangle is $\sqrt{3}(1-x^2)$. Thus the volume is

$$\int_{-1}^1 \sqrt{3}(1-x^2)\,dx = \sqrt{3}x - \frac{\sqrt{3}x^3}{3}|_{-1}^1 = \left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) - \left(-\sqrt{3} - \frac{-\sqrt{3}}{3}\right) = \frac{4\sqrt{3}}{3}.$$

These problems are sometimes known as volumes of "solids of rotation," because this technique is particularly good at solving problems like the following:

Example 6.31. What is the volume of the solid obtained by rotating the region bounded by $y = x^2, x = 5, y = 0$ about the x-axis?

We draw a picture, and see that the region has height x^2 at a point x, and thus the solid has a cross-section which is a circle of radius x^2, and thus an area of $\pi(x^2)^2$. It's clear that x varies from 0 to 5. So

$$V = \int_0^5 \pi x^4\,dx = \frac{\pi x^5}{5}|_0^5 = 5^4\pi - 0 = 625\pi.$$

Example 6.32. What is the volume of the solid obtained by rotatin the region bounded by $y = x^2, y = 25$ with $x \geq 0$ around the y-axis?

As before, we draw a picture. Our region has width $\sqrt{y}$ at a point y, and thus has cross-sectional area πy. Then y varies from 0 to 25, and the volume is

$$V = \int_0^{25} \pi y\,dy = \frac{\pi y^2}{2}|_0^2 5 = \frac{625\pi}{2}.$$

Note that in these problems it's easy to see which way to take our "slices": we want to get the circular cross-sections from the rotation, so we slice accordingly, and integrate along the axis we rotate around.

If our region touches the axis we rotate it around, these problems are straightforward: the cross-sectional area is the height (or width!) of the region squared times π. The problem is trickier if we have a hollow inside. We can still compute the cross-sectional area; it is the area of a *washer*, a circle with a smaller circle cut out of the center.

Remark 6.33. If a washer has outer radius R and inner radius r, then the area is $\pi R^2 - \pi r^2$, the area of the outer circle minus the radius of the inner.

Example 6.34. What is the volume of the solid given by rotating the region bounded by $y = x^2$ and $y = x$ around the x-axis.

At a point x, the cross-section of this solid is a washer. The outer circle has radius x and the inner circle has radius x^2, and thus the area of the cross-section is $\pi x^2 - \pi x^4$. So the volume is

$$V = \int_0^1 (\pi x^2 - \pi x^4)\,dx = \frac{\pi x^3}{3} - \frac{\pi x^5}{5} = \frac{\pi}{3} - \frac{\pi}{5} = \frac{2\pi}{15}.$$

We often find ourselves rotating these regions around lines other than the x- or y-axes. In this case we have to use our geometric intuition a bit more to sort out our cross-sectional areas.

Example 6.35. Rotate the same region about $y = 2$. We draw a picture; we see that we will get a solid whose cross-sections are washers centered at $y = 2$. The outer radius will be $2 - x^2$ and the inner radius will be $2 - x$, so the volume is

$$V = \int_0^1 \pi(2-x^2)^2 - \pi(2-x)^2\,dx = \pi\int_0^1 4 - 4x^2 + x^4 - 4 + 4x - x^2\,dx = \pi\int_0^1 x^4 - 5x^2 + 4x\,dx$$
$$= \pi\left(\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2\right)\Big|_0^1 = \pi(1/5 - 5/3 + 2) = \frac{4\pi}{15}.$$

Example 6.36. Find the volume of the solid generated by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ about the line $y = 1$.

We will integrate with respect to x since we rotate about a line parallel to the x-axis. We see that the curves intersect at $x = y = 0$ and $x = y = 1$. Our cross-sections are washers, and we see the outer radius is $1 - x$ and the inner radius is $1 - \sqrt{x}$. So the volume is

$$V = \pi\int_0^1 (1-x)^2 - (1-\sqrt{x})^2\,dx = \pi\int_0^1 x^2 - 3x + 2\sqrt{x}\,dx$$
$$= \pi\left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{4}{3}x^{3/2}\right)\Big|_0^1 = \pi\left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3}\right) = \frac{\pi}{6}.$$

6.5 Bonus material: Finding Volumes with Cylindrical Shells

Recall we want to find the volume of the solid obtained by rotating the region bounded by $x = 1, y = 2, y = \ln x$ about the x-axis. Slicing it into washers as before generates a difficult integral, so we will try to slice it a different way, by slicing it into *cylindrical shells*.

A cylindrical shell is what we get when we take a cylinder and remove a slightly smaller cylinder from the inside. If the outer radius is r_2 and the inner radius is r_1, it's not hard to see that the volume of the shell is $\pi r_2^2 h - \pi r_1^2 h = \pi h(r_2^2 - r_1^2)$. Less obviously, we factor $r_2^2 - r_1^2 = (r_2 + r_1)(r_2 - r_1)$ and write that the volume is $2\pi \frac{r_1+r_2}{2} h(r_2 - r_1) \approx 2\pi r h \Delta r$.

In many solids of rotation, we can slice the solid into a collection of cylindrical shells to approximate the volume, where the height of each cylinder is $f(x)$ for some x. We get the formula

$$V \approx \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x.$$

As before, our approximation gets better as we use more and thinner cylinders, and when we take the limit, we get

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x = \int_a^b 1\pi x f(x)\, dx,$$

where a is the inner radius of our entire solid, and b is the outer radius of the entire solid. (Note that this formula is essentially the surface area of the cylinder; this isn't an accident).

So for our earlier example, we can slice into cylinders whose height is in the x-direction. We see that

$$V = \int_0^2 2\pi y(e^y - 1)\, dy = 2\pi \left(ye^y - e^y - \frac{y^2}{2} \right) \Big|_0^2 = 2\pi(e^2 - 1).$$

Remark 6.37. Unlike in the method of washers, this time we will typically integrate with respect to x when we rotate around the y-axis, and vice versa.

Example 6.38. Find the volume of the solid obtained by rotating the region bounded by $y = 0$ and $y = x - x^2$ around the line $x = 2$.

Inverting the function $y = x - x^2$ would be a huge pain; so we'd like to integrate with respect to x, and thus use the cylinder method. Note that in this case the radius r is not x, but is $2 - x$. So the volume is

$$V = \int_0^1 2\pi(2-x)(x-x^2)\, dx = 2\pi \int_0^1 2x - 3x^2 + x^3\, dx = 2\pi \left(\frac{x^4}{4} - x^3 + x^2 \right) \Big|_0^1 = 2\pi(1/4 - 1 + 1) = \frac{\pi}{2}.$$

Example 6.39. What is the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 0, x = 1$ around the line $x = 1$?

$$V = \int_0^1 2\pi(1-x)x^3\,dx = 2\pi\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = 2\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{\pi}{10}.$$

Example 6.40. What is the volume of the solid obtained by rotating the same region around the line $x = 4$?

$$V = \int_0^1 2\pi(4-x)x^3\,dx = 2\pi\left(x^4 - \frac{x^5}{5}\right)\Big|_0^1 = 2\pi\left(1 - \frac{1}{5}\right) = \frac{8\pi}{5}.$$

Example 6.41. What is the volume of the solid obtained by rotating the region bounded by $xy = 1, x = 0, y = 1, y = 3$ about the x-axis?

We draw a picture, and conclude that to use the method of washers we'd have to break the region up into two pieces. Instead we integrate with respect to y and use cylindrical shells. We have y varying from 1 to 3, and the "height" of each cylinder is $1/y - 0$. So the volume is

$$V = \int_1^3 2\pi y(1/y)\,dy = \int_1^3 2\pi\,dy = 2\pi y|_1^3 = 4\pi.$$

Example 6.42. A word has to be said at this point about finding the volume of a sphere. We can view the sphere as a solid of rotation and find its volume using cross-sections:

$$\begin{aligned} V &= \int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2\,dx = \pi\int_{-r}^{r} r^2 - x^2\,dx = \pi\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r} \\ &= \pi\left(\left(r^3 - r^3/3\right) - \left(-r^3 + r^3/3\right)\right) = 4\pi r^3/3. \end{aligned}$$

But we can actually use another approach, similar in spirit to the method of cylindrical shells. We can look at the sphere as being made up of a collection of spherical shells. Taking inspiration from the cylindrical shells method, we see that the volume of each spherical shell will be "about" the surface area of the sphere times thickness; so we integrate the surface area of a sphere of radius x, as x varies from 0 to r. We get

$$V = \int_0^r 4\pi x^2\,dx = \frac{4\pi x^3}{3}|_0^r = \frac{4\pi r^3}{3}.$$

We haven't entirely justified our argument, but with more care we certainly could.